Math 1151, Exam 3 (in-class)
April 30, 2010

Name: *Amy's Solutions*

Discussion Section: *N/A*

Discussion TA: *N/A*

This exam has 8 multiple-choice problems, each worth 5 points. When you have decided on a correct answer to a given question, circle the answer in this booklet. There is no partial credit for the multiple-choice problems. This exam has 4 open-ended problems, whose point-values are given in the problem. Make sure to show all your work and circle your final answer. This exam is closed book and closed notes. You may use a scientific calculator but not a graphing calculator.

Formulas:

$$\sum_{k=1}^{n} \big(a_1 + (k-1)\,d\big) \;=\; \frac{n}{2}(a_1 + a_n)$$

$$\sum_{k=1}^{n} a_1\, r^{k-1} \;=\; a_1 \left(\frac{1-r^n}{1-r}\right)$$

1. For the vector $v = 3\hat{\imath} - 3\sqrt{3}\hat{\jmath}$, what is $\hat{v}$?

First compute the magnitude:

$$|v| \;=\; \sqrt{(3)^2 + (-3\sqrt{3})^2} \;=\; \sqrt{9+27} \;=\; \sqrt{36} = 6$$

Then divide v by $|v|$ to get the unit vector:

$$\hat{v} \;=\; \frac{v}{|v|} \;=\; \frac{3}{6}\,\hat{\imath} - \frac{3\sqrt{3}}{6}\,\hat{\jmath} \;=\; \frac{1}{2}\,\hat{\imath} - \frac{\sqrt{3}}{2}\,\hat{\jmath}$$

2. Find the equation for the parabola with focus $(4,0)$ and directrix $x = -4$.

This parabola opens to the right, and its vertex is at the origin, so its equation is of the form $y^2 = 4ax$. The distance from the focus to the vertex is $a = 4$, so the equation is $y^2 = 16x$.

3. Find the vertices of the hyperbola

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

This hyperbola opens up and down, and its center is at the origin, so the vertices are on the y-axis, and we get the coordinates by looking at the square root of the number under the y^2: $(0, \pm 3)$.

4. What is the value of the sum $\displaystyle\sum_{k=1}^{5} \big(2k+3\big)$?

Use linearity:

$$\sum_{k=1}^{5} \big(2k+3\big) \;=\; 2 \cdot \sum_{k=1}^{5} k \;+\; \sum_{k=1}^{5} 3 \;=\; 2 \cdot \left(\frac{5 \cdot 6}{2}\right) + 5 \cdot 3 \;= 45$$

Or use Gauss' trick:

$$\sum_{k=1}^{5} \big(2k+3\big) \;=\; \frac{5}{2}(5+13) \;= 45$$

5. Which best describes the following system of equations?

$$\begin{cases} 2x + 3y = 1 & (1) \\ -10x - 15y = -5 & (2) \end{cases}$$

Multiplying (1) by 5 and adding to (2) yields the equation $0 = 0$, which is trivially true ("duh" statement.) So the system has infinitely many solutions, i.e. it is a *consistent* system of *dependent* equations.

6. Which best describes the sequence $3, \frac{6}{5}, \frac{12}{25}, \frac{24}{125}, \ldots$?

This is a geometric series, because each term can be obtained from the one before it by multiplying by 2/5.

7. Find the sum: $4 + 11 + 18 + 25 + \ldots + 697$.

Notice that the numbers come from an arithmetic sequence with first term $a_1 = 4$ and common difference $d = 7$. So to get the sum we will add the first and the last, then multiply by $n/2$, where n is the index of the last term. To find the index of 697 we use the general formula:

$$a_n = a_1 + (n-1)d = 4 + 7(n-1) = 7n - 3$$

So if a_n is 697, that means

$$\begin{aligned} 7n - 3 &= 697 \\ 7n &= 700 \\ n &= 100 \end{aligned}$$

So the sum is

$$S_n = \frac{100}{2} \cdot (4 + 697) = 35,050$$

8. Find the sum: $\displaystyle\sum_{k=1}^{\infty} 5 \cdot \left(\frac{2}{3}\right)^{k-1}$.

This is a geometric series with $a_1 = 5$ and $r = 2/3$. Since $|r| < 1$, the series converges to

$$\frac{a_1}{1-r} = \frac{5}{1/3} = 15$$

9. (*10 points*) For the vectors $v = 2\hat{\imath} + 3\hat{\jmath}$, and $w = -\hat{\imath} + 3\hat{\jmath}$,

(a) Write v as the sum of two vectors v_1 and v_2, where v_1 is in the direction of w and v_2 is orthogonal to w.

The two vectors are

$$v_1 \;=\; \frac{v \cdot w}{|w|^2}\, w \qquad \text{and} \qquad v_2 \;=\; v - v_1$$

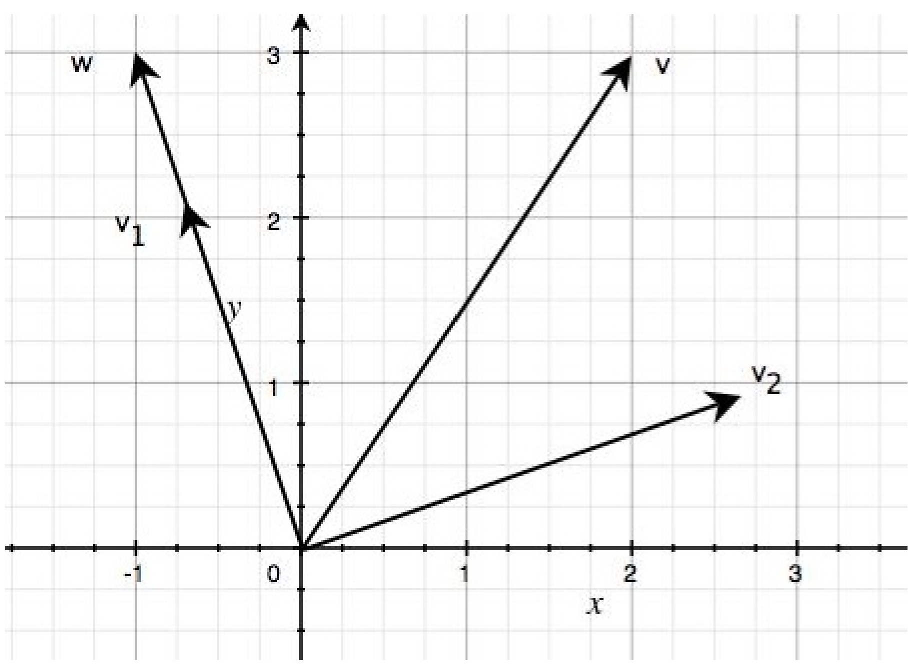
The dot product of v and w is $v \cdot w \;=\; (-2) + (9) \;=\; 7$, and the magnitude squared of w is $|w|^2 \;=\; (-1)^2 + (3)^2 \;=\; 10$. Plugging into the formula for v_1,

$$v_1 \;=\; \frac{7}{10}\, w \;=\; -\frac{7}{10}\,\hat{\imath} \;+\; \frac{21}{10}\,\hat{\jmath}$$

To get v_2 we just subtract this from v:

$$v_2 \;=\; (2 + \tfrac{7}{10})\,\hat{\imath} \;+\; (3 - \tfrac{21}{10})\,\hat{\jmath} \;=\; \tfrac{27}{10}\,\hat{\imath} + \tfrac{9}{10}\hat{\jmath}$$

(b) Graph v, v_1, v_2, and w on the same set of axes.



10. (*10 points*) For the conic section with the following equation,

$$4(x+2)^2 + 25(y-1)^2 = 100$$

(a) Find the center, foci, and vertices.

Divide both sides of the equation by 100 to get

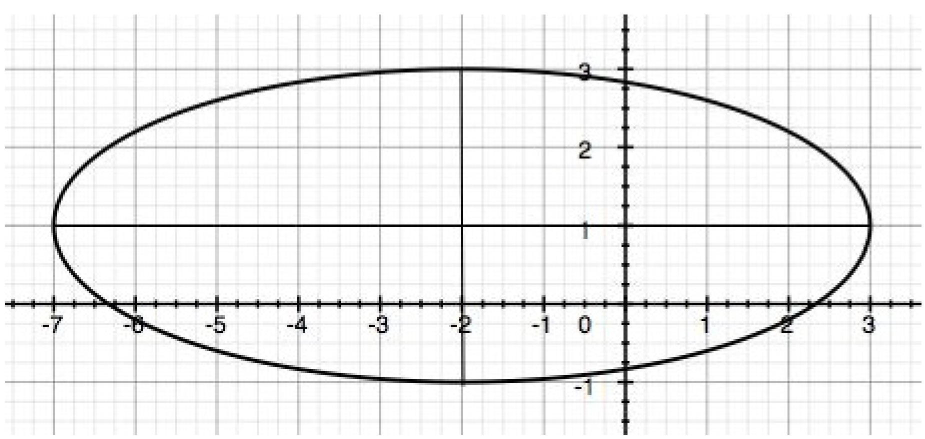
$$\frac{(x+2)^2}{25} + \frac{(y-1)^2}{4} = 1$$

Then we can see that this is an ellipse with center $(-2, 1)$. Since the larger number is underneath the x-term, the major axis is horizontal and $a = 5$, $b = 2$, and c is

$$c = \sqrt{a^2 - b^2} = \sqrt{21}$$

So the foci are $(-2 \pm \sqrt{21}, 1)$ and the vertices are $(-2 \pm 5, 1)$, i.e. $(-7, 1)$ and $(3, 1)$

(b) Graph the conic section.



11. (*20 points*) For $P(x) = x^3 - 3x^2 + 7x - 5$.

(a) List all the possible rational roots of $P(x)$.

Since the leading coefficient is $a_n = 1$, the possible rational roots are just the numbers that divide the constant term, which is $a_0 = -5$. So the possible rational roots are: ± 1, ± 5.

(b) Factor $P(x)$ over the real numbers.

Try plugging the possible rational roots in, starting with the easiest one, $x = 1$. We get

$$P(1) \;=\; 1 - 3 + 7 - 5 \;=\; 0$$

So $(x - 1)$ is a factor. We use polynomial long division to factor it out.

$$\begin{array}{r|l} & x^2 - 2x + 5 \\ \hline x - 1 & x^3 - 3x^2 + 7x - 5 \\ & -x^3 + x^2 \\ \hline & -2x^2 + 7x \\ & 2x^2 - 2x \\ \hline & 5x - 5 \\ & -5x + 5 \\ \hline & 0 \end{array}$$

Now we have to see if $x^2 - 2x + 5$ factors over the real numbers. There is no obvious way to factor it, so we use the quadratic formula to find the roots:

$$x \;=\; \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} \;=\; \frac{2 \pm \sqrt{-16}}{2} \;=\; \frac{2 \pm 4i}{2} \;=\; 1 \pm 2i$$

These roots are not real, so the quadratic $x^2 - 2x + 5$ does not factor over the real numbers. So, over the real numbers, $P(x)$ factors as

$$P(x) \;=\; (x - 1)(x^2 - 2x + 5)$$

(c) Factor $P(x)$ over the complex numbers.

As we computed above, the roots of $P(x)$ are 1, $1 + 2i$ and $1 - 2i$, so over the complex numbers $P(x)$ factors as

$$P(x) \;=\; (x - 1)(x - (1 + 2i))(x - (1 - 2i))$$

12. (*20 points*) Solve the system of equations:

$$\begin{cases} x + y - z = -1 & (1) \\ 4x - 3y + 2z = 16 & (2) \\ 2x - 2y - 3z = 5 & (3) \end{cases}$$

Is this system consistent or inconsistent? If consistent, are the equations dependent or independent?

Using matrices:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 4 & -3 & 2 & 16 \\ 2 & -2 & -3 & 5 \end{array}\right) \xrightarrow{R_2=-4r_1+r_2,\ R_3=-2r_1+r_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & -7 & 6 & 20 \\ 0 & -4 & -1 & 7 \end{array}\right) \xrightarrow{R_3=-r_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & -7 & 6 & 20 \\ 0 & 4 & 1 & -7 \end{array}\right)$$

$$\xrightarrow{R_2=2r_3+r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 1 & 8 & 6 \\ 0 & 4 & 1 & -7 \end{array}\right) \xrightarrow{R_3=-4r_2+r_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 1 & 8 & 6 \\ 0 & 0 & -31 & -31 \end{array}\right) \xrightarrow{R_3=-r_3/31} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 1 & 8 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_2=-8r_3+r_2,\ R_1=r_3+r_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1=r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \implies \begin{cases} x = 2 \\ y = -2 \\ z = 1 \end{cases}$$

So there is one solution, $(2, -2, 1)$, which means this is a *consistent* system of *independent* equations.

Scratch paper. (If you want your work on this page to be graded, make sure to label your work according to the problem you're solving, and make sure to write a note next to the original problem.)