**Math 1151, Polynomials, Factors, Zeros – Part II**
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## 1. Complex Roots

In the previous installment, we talked about a strategy for factoring polynomials: (1) Find a root $x = c$. (2) Use long division to factor out $(x - c)$. (3) Repeat. In particular, we can use the Rational Root Theorem to make educated guesses for the roots. The only problem with this strategy is that not every polynomial has *rational* roots. Remember the example $P(x) = x^2 - 2x - 1$. The quadratic formula tells us that its roots are *irrational*: $x = 1 \pm \sqrt{2}$. It factors as

$$P(x) = (x - (1 + \sqrt{2}))(x - (1 - \sqrt{2}))$$

We also know, from studying quadratics, that the roots might be *complex*. For example, $P(x) = x^2 - 6x + 25$ has roots

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(25)}}{2(1)} = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm \sqrt{-64}}{2} = 3 \pm 4i$$

In fact, the quadratic formula basically tells us, that if we allow complex numbers, every quadratic polynomial has two roots, i.e. it can be factored into linear factors. The truly remarkable thing is that allowing complex numbers is enough to ensure that we can factor *any* polynomial into linear factors, not just quadratics! Said another way, this means that, if wee allow complex numbers, any degree $n$ polynomial has $n$ roots. To put it simply:

Any polynomial factors into linear factors, if we allow complex roots.

This is a consequence of the Fundamental Theorem of Algebra, which is a significant piece of mathematics, and certainly beyond the scope of this class.

## 2. Complex Roots Occur in Conjugate Pairs

Notice that the complex roots that the quadratic formula gives us are *conjugate*. It turns out that complex roots always occur in conjugate pairs.

For a polynomial $P(x)$ with real coefficients, if $\alpha$ is a complex root of $P(x)$ then so is $\bar{\alpha}$.

The imaginary part cancels when we multiply $(x - \alpha)(x - \bar{\alpha})$, so the resulting polynomial has real coefficients. Let's see how this works for the example we had above:

$$\begin{aligned}(x - (3 + 4i))(x - (3 - 4i)) &= x^2 - (3 + 4i)x - (3 - 4i)x + (3 + 4i)(3 - 4i) \\ &= x^2 - 3x - 4ix - 3x + 4ix + 9 + 16 \\ &= x^2 - 6x + 25\end{aligned}$$

In general,

$$\begin{aligned}(x - \alpha)(x - \bar{\alpha}) &= x^2 - \alpha x - \bar{\alpha}x + \alpha\bar{\alpha} \\ &= x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha}\end{aligned}$$

Remember that the product $\alpha\bar{\alpha}$ is the modulus squared. If $\alpha = a + bi$,

$$\alpha\bar{\alpha} = (a + bi)(a - bi) = a^2 + abi - abi + b^2 = a^2 + b^2 = |\alpha|^2$$

so it is a real number. And when you add $\alpha$ and $\bar{\alpha}$, the imaginary parts cancel:

$$(a + bi) + (a - bi) = 2a$$

So multiplying $(x-\alpha)(x-\bar{\alpha})$ does give you a polynomial with real coefficients. In particular,

$$(x-\alpha)(x-\bar{\alpha}) = x^2 - 2ax + (a^2+b^2)$$

One other thing to note: since a polynomial will have as many roots as it's degree, as long as we allow complex roots, and since complex roots occur in conjugate pairs, that means that

An odd-degree polynomial must have at least one real root.

### 3. Worked Examples

**Example 1:** Suppose you know that $P(x)$ is a degree three polynomial with real coefficients, and you know that 4 and $3+i$ are zeros. What are the other zero(s)? Write $P(x)$ as a polynomial with real coefficients.

**Solution:**

We know that $P(x)$ will have exactly three roots, if we allow complex roots. We already have two, so there's just one left. Since one of the roots we're given, $3+i$, is complex, its conjugate $3-i$ must also be a root. To find $P(x)$ we multiply the linear factors:

$$P(x) = (x-4)(x-(3+i))(x-(3-i))$$

Remember the pattern we noticed above:

$$(x-\alpha)(x-\bar{\alpha}) = x^2 - 2ax + (a^2+b^2)$$

In this case, we get

$$P(x) = (x-4)(x^2-6x+10)$$

**Example 2:** Suppose you know that $P(x)$ is a degree six polynomial with real coefficients, and you know that $i$, $4-i$, and $2+i$ are zeros of $P(x)$. What are the other zeros? Write $P(x)$ as a polynomial with real coefficients.

**Solution:**

We know that $P(x)$ will have exactly six roots, as long as we allow complex roots. We already have three. Since those three are all complex roots, their conjugates must also be roots, so $-i$,, $4+i$ and $2-i$ are roots. That makes six. To find $P(x)$ we multiply the linear factors:

$$P(x) = (x-i)(x+i)(x-(4-i))(x-(4+i))(x-(2+i))(x-(2-i))$$

We multiply these out in pairs according to the pattern we noticed above:

$$\begin{aligned} (x-i)(x+i) &= x^2 - 2(0)x + 1 = x^2 - 1 \\ (x-(4-i))(x-(4+i)) &= x^2 - 2(4)x + (4^2+1^2) = x^2 - 8x + 5 \\ (x-(2+i))(x-(2-i)) &= x^2 - 2(2) + (2^2+1^2) = x^2 - 4x + 5 \end{aligned}$$

Putting them back together:

$$P(x) = (x^2-1)(x^2-8x+5)(x^2-4x+5)$$

**Example 3:** Given that $-5i$ is a zero of $P(x) = x^4 + 2x^3 + 30x^2 + 50x + 125$, find the remaining zeros.

**Solution:**

This polynomial should have four zeros, as long as we allow complex roots. Since $-5i$ is a root, its conjugate, $+5i$, is another root. So we can factor out $(x - 5i)(x + 5i) = x^2 + 25$ from $P(x)$.

$$
\begin{array}{r|l}
 & x^2 + 2x + 5 \\
\hline
x^2 + 25 & x^4 + 2x^3 + 30x^2 + 50x + 125 \\
 & -x^4 \qquad\quad - 25x^2 \\
\hline
 & 2x^3 + 5x^2 + 50x \\
 & -2x^3 \qquad\quad - 50x \\
\hline
 & 5x^2 \qquad + 125 \\
 & -5x^2 \qquad - 125 \\
\hline
 & 0
\end{array}
$$

The factor that is left is $x^2 + 2x + 5$. We use the quadratic formula to find the roots:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

So the zeros of $P(x)$ are: $\pm 5i, -1 \pm 2i$.

**Example 4:** Find the real and complex zeros of $P(x) = x^3 + 13x^2 + 57x + 85$, and write as a product of linear factors.

**Solution:**

Since $P(x)$ is of *odd* degree, it must have at least one real root. We use the rational root theorem to make an educated guess as to what that root might be. Since the leading coefficient is $a_3 = 1$ and the constant term is $a_0 = 85$, the possible rational roots are: $\pm 1, \pm 5, \pm 17, \pm 85$. Notice that,. in this case, plugging in a positive number will always yield a positive output, so we only need to check the negative numbers. (Start with the smaller ones and work up.)

$$
\begin{aligned}
P(-1) &= -1 + 13 - 57 + 85 \neq 0 \\
P(-5) &= -125 + 325 - 285 + 85 = 0
\end{aligned}
$$

So $(x + 5$ is a factor. Use long division:

$$
\begin{array}{r|l}
 & x^2 + 8x + 17 \\
\hline
x + 5 & x^3 + 13x^2 + 57x + 85 \\
 & -x^3 - 5x^2 \\
\hline
 & 8x^2 + 57x \\
 & -8x^2 - 40x \\
\hline
 & 17x + 85 \\
 & -17x - 85 \\
\hline
 & 0
\end{array}
$$

The quadratic that is left is $x^2 + 8x + 17$. This cannot be factored by hand, so we use the quadratic formula:

$$x = \frac{-8 \pm \sqrt{8^2 - 4(1)(17)}}{2(1)} = \frac{-8 \pm \sqrt{-4}}{2} = -4 \pm i$$

So the zeros of $P(x)$ are $-5, -4 + i, -4 - i$ and we can factor $P(x)$ as

$$P(x) = (x + 5)(x^2 + 8x + 17) = (x + 5)(x - (-4 + i))(x - (-4 - i))$$