Math 1151, Polynomials, Factors, Zeros – Part I
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1. Polynomials in Factored Form

Consider a polynomial in "factored form," for example

$$P(x) = (x + 17)(x + 1)(x - 2)(x - 5)$$

You can tell, just from looking at it, that it's zeros are $x = -17, -1, 2, 5$, because if you plug in one of those numbers, one of the factors will be zero, so the product will be zero. So,

> If a polynomial has a linear factor $(x - c)$, then $x = c$ is a zero, i.e. $f(c) = 0$.

It turns out that the converse is also true, i.e. if $f(c) = 0$, then $(x - c)$ is a factor. We will see why in a minute.

Before moving on, let's notice a few other things about this example. Notice that we can multiply it out to put it in "standard form,"

$$P(x) \ = \ a_n x^n \ + \ a_{n-1}x^{n-1} \ + \ \ldots + a_0$$

The number n is called the *degree* of $P(x)$. In our case, since there are four x's, the degree is four. The coefficient a_n is called the *leading coefficient* and the last term a_0 is called the *constant term*. Notice that in our case, the leading coefficient will be $a_4 = 1$, and the constant term will be $a_0 = (17)(1)(-2)(-5)$.

2. Polynomials in Standard Form

What about a polynomial in "standard form"? For example,

$$P(x) = x^3 - 2x^2 - x + 2$$

Can we factor this? Notice that the constant term is $a_0 = 2$. So perhaps $(x + 2)$ is a factor. Let's use polynomial long division to check:

$$
\begin{array}{r}
x^2 - 4x \quad + 7 \\
x + 2 \overline{) \quad x^3 - 2x^2 \quad - x \quad + 2} \\
\underline{- x^3 - 2x^2} \qquad\qquad \\
- 4x^2 \quad - x \qquad \\
\underline{4x^2 + 8x} \qquad \\
7x \quad + 2 \\
\underline{- 7x - 14} \\
- 12
\end{array}
$$

That means

$$P(x) = (x + 2)(x^2 - 4x + 7) - 12$$

Notice that there is a remainder of $R = -12$, so $(x + 2)$ is *not* a factor. Let's take another guess. Perhaps $(x - 2)$ is a factor.

$$
\begin{array}{r}
x^2 \qquad - 1 \\
x - 2 \overline{) \quad x^3 - 2x^2 - x + 2} \\
\underline{- x^3 + 2x^2} \qquad\quad \\
- x + 2 \\
\underline{x - 2} \\
0
\end{array}
$$

So $P(x) = (x-2)(x^2-1)$ with *zero as the remainder*. So $(x-2)$ is a factor. We can now see how to factor $P(x)$ completely,

$$P(x) \ = \ (x-2)(x^2-1) \ = \ (x-2)(x-1)(x+1)$$

The roots are $x = -1, 1, 2$.

3. The Remainder Theorem and the Factor Theorem

In general, if we divide a polynomial $P(x)$ by a linear factor $(x-c)$, there will be a remainder R, i.e.

$$P(x) = (x-c)Q(x) + R$$

and $(x-c)$ is a factor of $P(x)$ when the remainder is zero: $R = 0$. To see what the remainder will be, plug in $x = c$ to both sides of the equation.

$$P(c) \ = \ (c-c)Q(x) + R \ = \ 0 + R \ = \ R$$

So the remainder is exactly $P(c)$. (This result is called the "Remainder Theorem.") So $(x-c)$ is a factor of $P(x)$ when $P(c) = 0$. Combining this fact with the observation above, we can conclude that

$$(x-c) \text{ is a factor of } P(x) \text{ if and only if } P(c) = 0$$

This is called the "Factor Theorem." Basically, it says that finding roots is the same thing as finding linear factors. Notice that this means that there can't be more roots than the degree of the polynomial, because there can't be more linear factors than the degree of the polynomial.

4. Guessing Roots

Given a polynomial, our goal is to find as many zeros as possible. Said another way, our goal is to factor the polynomial as much as possible. Our strategy will be to guess a root, factor it out, guess another root, factor it out, etc. So we need to have a way to make educated guesses for roots. This is exactly what the Rational Root Test does for us.

> **Rational Root Test:** For a polynomial in standard form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, the only possible rational roots must have:
>
> numerators dividing the constant term a_0 (up to $\pm$)
>
> denominators dividing the leading coefficient a_n (up to $\pm$)
>
> In particular, if the polynomial is $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, the only possible rational roots are (± 1) times the factors of a_0.

Example: $P(x) = x^5 + 17x^4 - 7x^3 + x^2 - 3x + 14$

> Any rational root must divide 14 evenly (up to $\pm$). So the possible rational roots are $\pm 1, \pm 2, \pm 7$.

Example: $P(x) = 6x^6 + 2x^4 + 7$

> Any rational root must have a numerator dividing 7 (up to $\pm$) and a denominator dividing 6 (up to $\pm$).
>
> possible numerators: $\pm 1, \pm 7$
>
> possible denominators: $\pm 1, \pm 2, \pm 3, \pm 6$

So the possible rational roots are:

$$\pm 1, \quad \pm\tfrac{1}{2}, \quad \pm\tfrac{1}{3}, \quad \pm\tfrac{1}{6}$$

$$\pm 7, \quad \pm\tfrac{7}{2}, \quad \pm\tfrac{7}{3}, \quad \pm\tfrac{7}{6}$$

5. Bottom line

We use the following strategy to try to factor a polynomial completely.

1. Look at the degree to see how many roots there could be.
2. Use the Rational Root Test to try to guess a root.
3. Check to see if it is a root by plugging in.
4. If it is a root, factor it out, and use long division of polynomials to see what's left.
5. Look at the polynomial that's left after long division, and try to guess a root of that polynomial ...

Repeat this procedure until all you have left is a quadratic. (We know how to deal with quadratics!)

Example: Find all the real zeros of $P(x) = x^3 + 8x^2 + 11x - 20$ and factor it out.

Following our strategy:

1. The degree is three, so there are at most three real roots.
2. Since the leading coefficient is $a_3 = 1$ and the constant term is $a_0 = 20$, the possible rational roots must be integers that divide 20 (up to $\pm$), i.e. $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$. We can pick any number on this list as our first guess. Let's start with the easiest one: $x = 1$.
3. Check to see if $x = 1$ is a root by plugging in:

$$P(1) = (1)^3 + 8(1)^2 + 11(1) - 20 = 0$$

So $x = 1$ is a root.

4. Factor out $(x - 1)$ using long division:

$$\begin{array}{r|l} & \phantom{x^3 + {}} x^2 + 9x + 20 \\ \hline x - 1 & x^3 + 8x^2 + 11x - 20 \\ & -x^3 + x^2 \\ & \phantom{x^3 +{}} 9x^2 + 11x \\ & \phantom{x^3 +{}} -9x^2 + 9x \\ & \phantom{x^3 + 9x^2 +{}} 20x - 20 \\ & \phantom{x^3 + 9x^2 +{}} -20x + 20 \\ & \phantom{x^3 + 9x^2 + 20x - 2{}} 0 \end{array}$$

5. What we have left is the quadratic $x^2 + 9x + 20$, and we can factor this:

$$x^2 + 9x + 20 = (x + 4)(x + 5)$$

So we can conclude that $P(x) = (x - 1)(x + 4)(x + 5)$ and the real roots are $x = 1$, $x = -4$, $x = -5$.

6. Watch Out!

Our strategy is a good one, but it is not guaranteed to find all the real roots, because not all real roots are *rational*. We have already encountered this fact in finding the roots of quadratics. Sometimes you can factor quadratics by hand, but sometimes you have to use the quadratic formula because the roots have square roots in them.

For example, $P(x) = x^2 - 2x - 1$. The possible rational roots are ± 1, but if you plug those in you don't get zero. If we use the quadratic formula,

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

So $P(x)$ factors as $P(x) = (x - (1 + \sqrt{2}))(x - (1 - \sqrt{2}))$.

So it's not hard to make examples of polynomials that don't have any rational roots. For example, if we multiply two quadratics that don't have rational roots, we'd get a degree four polynomial with no rational roots. In this case our strategy will not enable us to find the roots.