

The Basics of Exponentials and Logarithms

Math 1271, TA: Amy DeCelles

This is a summary of the basic facts you need to know about exponential and logarithmic functions. The material is from sections 1.5 and 1.6.

1. Laws of Exponents

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = \frac{a^x}{a^y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

2. Exponential Growth and Decay

Exponential functions come up frequently in modeling real world phenomena. Examples include population growth and radioactive decay. Exponential functions are characterized in the following way: given a certain unit of time (say one day) the amount of something (say the number of bacteria in a petrie dish) will increase (or decrease) by the same factor or ratio. So in the case of bacteria, we might say that the population doubles each day. This is exponential growth. If we start with 100 bacteria, the amount of bacteria would be modeled by the following equation:

$$P(t) = 100 \cdot 2^t$$

where $P(t)$ is the number of bacteria after t days.

Exponential decay is the reverse: the amount of a certain radioactive substance decreases by the same ratio every year, say by one-half. Then if we start with 100 grams, the next year there will be 50 grams, then 25 grams, etc. Here the amount of radioactive substance would be modeled by

$$A(t) = 100 \cdot (1/2)^t$$

where $A(t)$ is the amount of radioactive substance left after t years.

3. The Graphs

There are a few things to notice about exponential functions $f(x) = a^x$ from their graphs:

1. They have only positive outputs.
2. The line $y = 0$ is a horizontal asymptote.
3. If $a > 1$, the function is increasing constantly; if $0 < a < 1$ it is decreasing constantly.

4. The number e

Looking at graphs of exponential functions $f(x) = a^x$, we can quickly see that all of them intersect the y -axis at $y = 1$ (since $a^0 = 1$, for every $a \neq 0$.) However the angle at which the graph hits the y -axis changes. We might wonder when it is that the graph will hit the y -axis with slope exactly equal to 1. This occurs for a particular number, which we call e . The number e is transcendental

(like π), but it is approximately 2.71828. The thing to remember is that even though e is represented by a letter, it is a *constant number* not a *variable*.

5. Inverse Functions

Every one-to-one function has an inverse function, i.e. a function that switches inputs and outputs. (You can tell that a function is one-to-one if it passes the horizontal line test.) Inverses *undo* each other, i.e. if we input x to a function f , and get y as an output, then we input y into the inverse function f^{-1} of f , then we get x back as the output; we're back where we started. In other words: if $f(x) = y$, then $f^{-1}(y) = x$.

If you have a one-to-one function f , you can find its inverse by:

1. Let $y = f(x)$.
2. Solve for x .
3. Switch x and y .
4. Write $f^{-1}(x) = y$.

The graph of f^{-1} is obtained by reflecting the graph of f through the line $y = x$. (You're basically just switching the role of x and y .)

6. Logarithms

Logarithms are the inverse functions for exponential functions. For a number $a \neq 0$, we define the logarithmic function with base a by:

$$\log_a(x) = y \text{ if and only if } a^y = x.$$

The natural logarithm, denoted \ln , is the logarithm base e . Notice that since the exponential functions always have positive outputs, the logarithms only make sense for positive inputs.

7. Laws of Logarithms

For x, y positive real numbers, and r any real number,

1. $\log_a(xy) = \log_a(x) + \log_a(y)$
2. $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
3. $\log_a(x^r) = r \log_a(x)$

Notice that these match up with the first three laws of exponents.

Another fact that comes in handy is the change of base formula:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$