2.1 The Tangent and Velocity Problems

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1. Overview

The Tangent Problem:

Let's say you have a graph of a function. If you were feeling ambitious you might have the desire to find a line that touches the graph at a certain point, hitting it at just the right angle: so that the slope of the line matches the slope of the graph at that point. How could we do this? We can calculate the slopes of secant lines to approximate the slope of the tangent.

The Velocity Problem:

What's the definition of velocity (or speed)? Change in position over change in time, right? But now think about this: When you're driving down the highway, you can look at your spedometer at any moment, and it will tell you how fast you're going at that moment. Tell me, what's the change in time? Zero? We need a new definition of velocity to capture the meaning we intuitively know is there: instantaneous velocity (vs average velocity.) We can calculate average velocities (with smaller and smaller time intervals) to approximate the instantaneous velocity.

2. Examples

1.) Suppose you are hooked up to a machine that counts your heartbeats, and it tells you that after 36 minutes, you have had 2530 heartbeats, after 38 minutes you have had 2661 heartbeats, and so on as recorded in the following table:

t (min)	36	38	40	42	44
heartbeats	2530	2661	2806	2948	3080

Estimate your heartrate after 42 minutes using the following time intervals:

(a) $t = 36$ to $t = 42$

(b) $t = 38$ to $t = 42$

(c) $t = 40$ to $t = 42$

(d) $t = 42$ to $t = 44$

Evaluate your estimates (which are better, which are worse).
Solution:

The heartrate is the number of heartbeats per minute. So to estimate the heartrate at 42 minutes we will look at the intervals of time given to us in (a)-(d) above, and divide the number of heartbeats by the number of minutes.

So for (a), the number of heartbeats in the time interval is $2948 - 2530$ and the number of minutes is $42 - 36$, so we estimate the heartrate to be:

$$\frac{2948 - 2530}{42 - 36} = 70 \quad \text{heartbeats per minute}$$

For (b), we estimate the heartrate to be:

$$\frac{2948 - 2661}{42 - 38} = 72 \quad \text{heartbeats per minute}$$

For (c), we estimate the heartrate to be:

$$\frac{2948 - 2806}{42 - 40} = 71 \quad \text{heartbeats per minute}$$

For (d), we estimate the heartrate to be:

$$\frac{3080 - 2948}{44 - 42} = 66 \quad \text{heartbeats per minute}$$

If you graph the data points, with time on the x-axis and heartbeats on the y-axis, you can see that the estimations that we made for the heartrate are slopes of secant lines. The actual heartrate would be the slope of the *tangent* line to the graph at $t = 42$. So the best estimates would be the secant lines in (c) and (d). To get an even better estimate we could take an average of the two: 68.5 heartbeats per minute.

2.) Suppose an astronaut standing on the moon threw a candy bar with an initial velocity of 53 meters per second. The height of the candy bar is then given by the following equation:

$$h(t) = 58t - 8.3t^2$$

where t is the number of seconds after the astronaut threw the candy bar.

(a) Find the average velocity over the given time intervals:

- (i) [1,2]
- (ii) [1, 1.5]
- (iii) [1, 1.1]
- (iv) [0,1]
- (v) [0.5, 1]
- (vi) [0.9, 1]

(b) Use your computations from (a) to guess what the velocity of the candy bar is when $t = 1$.

Solution:

For (a) we need to know that the average velocity is just displacement over time:

$$v_{\text{ave}} = \frac{\Delta h}{\Delta t}$$

So for each time interval we need to find Δh and Δt.

(i) For the time interval [1,2], the displacement is:

$$\Delta h = h(2) - h(1) = (58 \cdot 2 - .83 \cdot 2^2) - (58 \cdot 1 - .83 \cdot 1) = 55.48$$

The change in time is just 1, so the average velocity is:

$$v_{\text{ave}} = \frac{55.48}{1} = 55.48 \quad \text{meters per second}$$

(ii) For the time interval [1, 1.5], the displacement is:

$$\Delta h = h(1.5) - h(1) = (58 \cdot 1.5 - .83 \cdot (1.5)^2) - (58 \cdot 1 - .83 \cdot 1) = 27.95$$

The change in time is .5 seconds, so the average velocity is:

$$v_{\text{ave}} = \frac{27.95}{.5} = 55.91 \quad \text{meters per second}$$

(iii) For the time interval [1, 1.1], the displacement is:

$$\Delta h = h(1.1) - h(1) = (58 \cdot 1.1 - .83 \cdot (1.1)^2) - (58 \cdot 1 - .83 \cdot 1) = 5.624$$

The change in time is .1 seconds, so the average velocity is:

$$v_{\text{ave}} = \frac{5.624}{.1} = 56.24 \quad \text{meters per second}$$

(iv) For the time interval [0, 1], the displacement is:

$$\Delta h = h(1) - h(0) = (58 \cdot 1 - .83 \cdot 1) - (58 \cdot 0 - .83 \cdot 0) = 57.16$$

The change in time is 1 second, so the average velocity is:

$$v_{\text{ave}} = \frac{57.16}{.5} = 57.16 \quad \text{meters per second}$$

(v) For the time interval [.5, 1], the displacement is:

$$\Delta h = h(1) - h(.5) = (58 \cdot 1 - .83 \cdot 1) - (58 \cdot .5 - .83 \cdot (.5)^2) = 28.37$$

The change in time is .5 seconds, so the average velocity is:

$$v_{\text{ave}} = \frac{28.37}{.5} = 56.74 \quad \text{meters per second}$$

(vi) For the time interval [.9, 1], the displacement is:

$$\Delta h = h(1) - h(.9) = (58 \cdot 1 - .83 \cdot 1) - (58 \cdot .9 - .83 \cdot (.9)^2) = 5.641$$

The change in time is .1 seconds, so the average velocity is:

$$v_{\text{ave}} = \frac{5.641}{.1} = 56.41 \quad \text{meters per second}$$

For part (b) we need to evaluate our estimates. The best estimates for the instantaneous velocity at $t = 1$ are (iii) and (vi), because those are the shortest time intervals. We can make our guess even better by taking the average of the two: 56.32 meters per second.