

2.2 The Limit of a Function

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1. Overview

The main points:

1. Definition of a limit, notation
2. One-sided limits, notation
3. Infinite limits, vertical asymptotes
4. Graphs

Things to remember:

1. The limit exists if and only if both one-sided limits exist and are equal.
2. Holes in the graph do not mess up the limit, but jumps and vertical asymptotes do.
3. Vertical asymptotes are *lines* (not numbers).

2. Examples

1.) Calculate the following infinite limit:

$$\lim_{x \rightarrow 5^-} \frac{6}{x - 5}$$

We are looking at a left-hand limit here, so we are trying to see what happens as x gets close to 5 from the left, i.e. $x < 5$. We look at the numerator and denominator separately:

	size	sign
numerator	6	+
denominator	small	-

Since the denominator is getting small, while the numerator stays at 6, that means that the fraction is getting *large*. (Think: $\frac{6}{1} = 6$, $\frac{6}{.1} = 60$, $\frac{6}{.01} = 600$, $\frac{6}{.001} = 6000$, ... getting larger and larger!) So this means that the limit is either $+\infty$ or $-\infty$. Since $x < 5$, the numerator is positive and the denominator is negative, which means that the fraction is negative. So the limit is:

$$\lim_{x \rightarrow 5^-} \frac{6}{x - 5} = -\infty$$

2.) Calculate the following infinite limit:

$$\lim_{x \rightarrow 3} \frac{2 - x^3}{(3 - x)^2}$$

We need to look at the numerator and denominator when x is close to 3:

	size	sign
numerator	≈ -6	-
denominator	small	+

Since the denominator is getting small, while the numerator gets close to a (non-zero, finite) number, that means that the value of the function is getting large. When x is close to 3, the numerator is negative and the denominator is positive, so the function is negative. So the limit is:

$$\lim_{x \rightarrow 3} \frac{2 - x^3}{(3 - x)^2} = -\infty$$

3.) Calculate the following infinite limit:

$$\lim_{x \rightarrow \pi^-} \csc x$$

You may or may not remember that cosecant has an asymptote at $x = \pi$, but a way to see that this is so is to remember that cosecant is the reciprocal of sine:

$$\csc x = \frac{1}{\sin x}$$

and $\sin \pi = 0$. Since the numerator is just 1 (a positive, finite number!) we just need to determine whether the denominator is positive or negative when x is approaching π from the left, i.e. x is close to π and $x < \pi$. If you remember your unit circle, you can see that the denominator will be positive, because sine is positive on the whole first quadrant. So to sum up what we've figured out:

	size	sign
numerator	1	+
denominator	small	+

So the limit is:

$$\lim_{x \rightarrow \pi^-} \csc x = \lim_{x \rightarrow \pi^-} \frac{1}{\sin x} = +\infty$$