2.3 Limit Laws

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1. Overview

Outline:

1. Sum, difference, constant multiple, product, quotient* of limit is limit of ...
   (* as long as the limit of the denominator is not zero)
2. Power law, root law
3. Two easy limits
4. Direct substitution property
5. Three theorems

Two easy limits:

$$\lim_{x \to a} c = c$$

$$\lim_{x \to a} x = a$$

Direct Substitution Property
If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Three theorems:

1. The limit of $f(x)$ as $x \to a$ equals L iff the limits from the left and right both equal L.
2. If $f(x) \leq g(x)$ near a (except possibly at a), then:

   $$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

3. Squeeze Theorem:
   If $f(x) \leq g(x) \leq h(x)$ near a (except possibly at a), and

   $$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

   then,

   $$\lim_{x \to a} g(x) = L$$

2. Examples

1.) Compute the following limit:

$$\lim_{x \to -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4}$$

The first thing you should do when trying to compute a limit is to *plug in the number*. If we plug in -4 to the function in this example we get:

$$\frac{16 - 20 + 4}{16 - 12 - 4}$$

Notice that the denominator would be zero (not allowed!) So we know that the function has a problem at $x = -4$: either a hole or an asymptote. Notice that the numerator would also be zero. This means that we should try the "factor and cancel" technique:

$$\begin{aligned}
\lim_{x \to -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} &= \lim_{x \to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} \\
&= \lim_{x \to -4} \frac{(x+1)}{(x-1)} \\
&= \frac{-4+1}{-4-1} \\
&= \frac{3}{5}
\end{aligned}$$

Since the "factor and cancel" technique worked, that means that the original function has a *hole* at $x = -4$.

NB Do not write:

$$\frac{(x+4)(x+1)}{(x+4)(x-1)} = \frac{(x+1)}{(x-1)}$$

That is true *if* $x \neq -4$, but you can't just say that those two functions are the same. The one on the left has a hole at $x = -4$, and the function on the right does *not* have a hole at $x = -4$.

2.) Compute the following limit:

$$\lim_{x \to -1} \frac{x^2 + 5x + 4}{x^2 + 3x - 4}$$

Again, the first thing to do is just plug the number in. This time we get:

$$\frac{1 - 5 + 4}{4 - 3 - 4} = \frac{0}{-3} = 0$$

While having zero in the denominator is a problem, there is *no problem* having zero in the numerator. So the function does not have any problems at $x = -1$. This means (because of the Direct Substitution Property) that the limit is equal to the function at that point:

$$\lim_{x \to -1} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \frac{1 - 5 + 4}{4 - 3 - 4} = \frac{0}{-3} = 0$$

3.) Compute the following limit:

$$\lim_{x \to -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$$

We plug in $x = -1$ and get:

$$\frac{1+4}{1+3-4}$$

The denominator would be zero, so we know that the function has a problem at $x = -1$, either a hole or an asymptote. Notice that the top would be $5 \neq 0$. This means that we *cannot* factor and cancel. We have an asymptote. So now we just have to determine the behavior at the asymptote. To do this we look at what happens as we approach from the left and from the right.

As $x \to -1^-$

- top $\to -3$ (negative)
- bottom $\to 0$ and is positive

So the limit from the left is $-\infty$.

As $x \to -1^+$

- top $\to -3$ (negative)
- bottom $\to 0$ and is negative

So the limit from the right is $+\infty$.

Since the limit from the right does not agree with the limit from the left, the limit does not exist.

4.) Compute the following limit:

$$\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h}$$

If we plug in $h = 0$ we get zero over zero, so we would like to do something like the "factor and cancel technique" used above, in order to eliminate the zero in the denominator. In this problem we can use a technique called *rationalizing the numerator*. We multiply top and bottom by the conjugate:

$$\begin{aligned}
\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} &= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \\
&= \lim_{h \to 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + 1)} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{1+h} + 1)}
\end{aligned}$$

Now we can cancel the h, which was the problem, and plug in $h = 0$:

$$\begin{aligned}
\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} &= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} \\
&= \frac{1}{\sqrt{1+0} + 1} \\
&= \frac{1}{2}
\end{aligned}$$

5.) If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

Notice that we don't know what $g(x)$ is. So hopefully that doesn't matter. What we *do* know about $g(x)$ is that it is sandwiched between two other functions: $2x$ and $x^4 - x^2 + 2$. So we hope that the Squeeze Theorem will help us out. Notice that:

$$\lim_{x\to 1} 2x = 2 \quad \text{and} \quad \lim_{x\to 1} x^4 - x^2 + 2 = 1 - 1 + 2 = 2$$

Since the two functions come together as $x \to 1$, and since $g(x)$ is always in between them, that means that $g(x)$ also approaches the same place as $x \to 1$, i.e.

$$\lim_{x\to 1} g(x) = 2$$

6.) Compute the following limit:

$$\lim_{x\to -2} \frac{2 - |x|}{2 + x}$$

If we plug in $x = -2$, we would get zero over zero, so hopefully we can use some trick to eliminate the zero in the denominator. The key to this problem is figuring out how to treat the absolute value. Notice that when x is *close* to -2, it is negative, so we can treat $|x|$ like $-x$ when we're computing the limit. (This might be confusing to you, but try thinking about an example like $-(-5) = |-5|$.) So we can compute:

$$\begin{aligned}
\lim_{x\to -2} \frac{2 - |x|}{2 + x} &= \lim_{x\to -2} \frac{2 - (-x)}{2 + x} \\
&= \lim_{x\to -2} \frac{2 + x}{2 + x} \\
&= \lim_{x\to -2} \frac{1}{1} \\
&= 1
\end{aligned}$$

If this problem is confusing to you, I suggest drawing a graph so you can see what's going on.