2.4 The Precise Definition of a Limit

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1. Overview

Definition of a Limit

We say "the limit of $f(x)$ as x approaches a is L" if the following condition is satisfied:

> For every number $\epsilon > 0$ there is a number $\delta > 0$ such that:
>
> $$\text{if} \quad |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Parsing this definition

Our intuitive understanding is that L is the limit (of $f(x)$ as $x \to a$) if $f(x)$ gets closer and closer to L as x gets closer and closer to a. Let's see how this matches up with the precise definition. First look at the expression $|x - a|$. This is the *distance* between x and a. So if we make δ smaller and smaller, that means that x is getting closer and closer to a. Similarly, $|f(x) - L|$ is the *distance* between the y-values $f(x)$ and L, so if ϵ gets smaller and smaller, that means that $f(x)$ is getting closer and closer to L. So, just to make this clear: δ is a distance that specifies an x-range: how far away from a can x be? ... it must be within δ units of a. We'll call this x-range a "δ-neighborhood" of a. And ϵ is a distance that specifies a y-range: how far away from L can $f(x)$ be? ... it will be within ϵ units of L. This y-range we'll call an ϵ-neighborhood of L.

So when we say that:

$$\text{if} \quad |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

that is like saying:

> If x is close enough to a (namely in a δ-neighborhood of a) then $f(x)$ is guaranteed to be close to L (namely in an ϵ-neighborhood of L).

Ok, well then, what is the deal with the "for every $\epsilon > 0$ there is a $\delta > 0$" part? This means that no matter how small you make the ϵ-neighborhood of L, you will always be able to find a δ-neighborhood of a, that "works," i.e. a δ-neighborhood small enough to guarantee that the y-values of the graph of f are in the ϵ-neighborhood of L. The point is that we can get $f(x)$ infinitely close to L, by just making the δ-neighborhood of a smaller and smaller.

2. Example

Problem: Prove, using the ϵ-δ definition of limit that:

$$\lim_{x \to 1} 5x - 3 = 2$$

Solution:

> We need to show:
>
> > For any $\epsilon > 0$, there is a $\delta > 0$ such that:
> >
> > $$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |(5x - 3) - 2| < \epsilon$$

Before we write our proof, we need to do some thinking. (This is like the prewriting you would do before writing a paper.) We treat ϵ like a fixed number. We want to figure out what δ will work, given the ϵ we have. We start with the ϵ condition:

$$|(5x-3)-2| < \epsilon$$

Simplifying, we get:

$$|5x-5| < \epsilon$$

We factor out a 5:

$$5 \cdot |x-1| < \epsilon$$

So if we rewrite what we have to show, using the simplification we just did, we get:

> For any $\epsilon > 0$, there is a $\delta > 0$ such that:
>
> $$\text{if} \quad |x-1| < \delta \quad \text{then} \quad 5 \cdot |x-1| < \epsilon$$

Well, if $|x-1| < \delta$ then $5 \cdot |x-1| < 5\delta$. So we just need to have $5\delta \leq \epsilon$. So we will choose $\delta = \frac{\epsilon}{5}$.

Now that we have figured out what δ will work, we need to go back and write up an argument. (This is like writing a paper: we take the work we just did and arrange it nicely to construct an argument.)

Proof:
Given any $\epsilon > 0$, we can define $\delta = \frac{\epsilon}{5}$. Then:

$$\begin{aligned} \text{if} |x-1| < \delta: \quad & \text{then} \quad |x-1| < \frac{\epsilon}{5} \\ & \text{then} \quad 5 \cdot |x-1| < 5 \cdot \frac{\epsilon}{5} \\ & \text{then} \quad 5 \cdot |x-1| < \epsilon \end{aligned}$$

But $5 \cdot |x-1| = |(5x-3)-2|$, so we have shown:

$$\text{if} \quad |x-1| < \delta \quad \text{then} \quad |(5x-3)-2| < \epsilon$$

So we have shown:

> For any $\epsilon > 0$, there is a $\delta > 0$ such that:
>
> $$\text{if} \quad |x-1| < \delta \quad \text{then} \quad |(5x-3)-2| < \epsilon$$

and we are done!