2.5 Continuity

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1. Overview

Outline:

1. Definition of continuity (implicitly has 3 parts)
2. Graph example
3. Some continuous functions
4. Intermediate Value Theorem

Definition of Continuity

A function f is continuous at a number a if:

$$\lim_{x \to a} f(x) = f(a)$$

Notice that this definition implicitly requires three things:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x)$ equals $f(a)$

Graph example

Given a graph of a function $f(x)$ and some specified x-values $x_0 = a, b, c, ...$ you should be able to:

1. Find $f(x_0)$ (if it exists).
2. Find $\lim_{x \to x_0} f(x)$ (if it exists).
3. Determine whether f is continuous at x_0.
4. Explain why f is/is not continuous at x_0.

Some continuous functions

The good news is that almost all the functions you know are continuous:

> **Theorem:** All polynomials, rational functions, root functions, trig functions, inverse trig functions, exponential functions, and logarithmic functions are continuous where defined.

Intermediate Value Theorem

> Suppose f is continuous on a closed interval $[a, b]$. Suppose the y-values $f(a)$ and $f(b)$ are not equal. Pick any number N between $f(a)$ and $f(b)$. Then there is a number c in (a, b) such that $f(c) = N$.

Notice that the IVT does *not* tell you *anything* about the number c, except that it is between a and b! So you should not expect to find c, unless you are *specifically* asked to find it.

2. Examples

1.) Use the definition of continuity and the limit laws to show that $f(x) = x^2 + \sqrt{7-x}$ is continuous at $a = 4$.

First of all, notice that we're supposed to use the *definition* of continuity in this problem. (In particuar that means that, for this problem, we can't argue that since power functions and root functions are continuous (where defined) and $f(4)$ is defined, then $f(x)$ is continuous at $a = 4$. Even though that is a correct argument, we are asked to do the problem a different, and yes, more tedious, way.)

Recall the definition of continuity: A function $f(x)$ is continuous at a if:

$$\lim_{x\to a} f(x) = f(a)$$

So in this problem we need to do three things:

1. Use the limit laws to evaluate the limit:

$$\lim_{x\to 4} \left(x^2 + \sqrt{7-x}\right)$$

2. Find $f(4)$.
3. Show that they are equal.

We will start with the limit. Remember the limit laws from 2.3 (sum law, difference law, product law, ...) When we are told to "evaluate the limit using the limit laws", that means that we have to show *every* step.

$$\begin{aligned}
\lim_{x\to 4} \left(x^2 + \sqrt{7-x}\right) &= \left(\lim_{x\to 4} x^2\right) + \left(\lim_{x\to 4} \sqrt{7-x}\right) \\
&= \left(\lim_{x\to 4} x\right)^2 + \sqrt{\lim_{x\to 4}(7-x)} \\
&= \left(\lim_{x\to 4} x\right)^2 + \sqrt{\lim_{x\to 4} 7 - \lim_{x\to 4} x} \\
&= (4)^2 + \sqrt{7-4} \\
&= 16 + \sqrt{3}
\end{aligned}$$

Now we find $f(4)$:

$$f(4) = (4)^2 + \sqrt{7-4} = 16 + \sqrt{3}$$

Since the two are equal, the function is continuous at $a = 4$.

2.) Use continuity to evaluate the limit:

$$\lim_{x\to 1} e^{x^2-1}$$

This problem is the opposite of the previous one. In the previous problem, we used limit laws to prove continuity. Now we are using continuity to find a limit. What this means is that we need to show that $f(x) = e^{x^2-1}$ is continuous at $a = 1$ (using a theorem, not using the definition of continuity.) Then we use the definition of continuity to say that the limit is equal to the value of the function at $a = 1$.

So what can we do to show that this function is continuous? First we notice that $f(x)$ is the composition of two functions: $h(x) = e^x$ and $g(x) = x^2 - 1$. Both of these functions are continuous everywhere (Theorem 7), so that means that $f(x)$ is continuous everywhere (Theorem 4). In particular $f(x)$ is continuous at $a = 1$. But that means that:

$$\lim_{x \to 1} f(x) = f(1)$$

So we can evaluate the limit:

$$\lim_{x \to 1} e^{x^2-1} = e^{1-1} = e^0 = 1$$

3.) Where is $f(x)$ continuous?

(a) $f(x) = \frac{1}{1-x}$

(b) $f(x) = x^5 + 3x + 2$

(c) $f(x) = \begin{cases} 3x & \text{if } x < 1 \\ (x-1)^2 + 3 & \text{if } x \geq 1 \end{cases}$

(d) $f(x) = \begin{cases} 3x & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

(e) $f(x) = \frac{1-x^2}{2-x-x^2}$

(f) $f(x) = \begin{cases} \frac{1-x^2}{2-x-x^2} & \text{if } x \neq 1 \\ \frac{2}{3} & \text{if } x = 1 \end{cases}$

For (a), $f(x)$ is a rational function, so it is continuous where defined, i.e. for all $x \neq 1$.

For (b), $f(x)$ is a polynomial function, so it is continuous where defined, i.e. for all real numbers.

For (c), $f(x)$ is a piecewise function, so we need to be more careful. We have to look at each piece and also look at the crossover point. Notice that both pieces of this function are polynomials: $3x$ and $(x-1)^2 + 3$. So each piece, considered separately, is continous. The only thing we need to check is the crossover point $x = 1$. We need plug $x = 1$ into each piece and see if they match. When we do that we are basically checking to see whether or not the left and right limits agree:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x-1)^2 + 3 = (1-1)^2 + 3 = 3$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3x = 3(1) = 3$$

So the pieces do match (meaning the left and right limits agree.) So the limit is:

$$\lim_{x \to 1} f(x) = 3$$

and the value of the function is also 3, because:

$$f(3) = 3(1) = 3$$

So, using the definition of continuity, $f(x)$ is continuous at $a = 3$.

For (d), we have another piecewise function. Each piece, considered separately, is continuous, so we just have to check the crossover point $x = 1$. We compute left and right limits as before:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3x = 3(1) = 3$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = (1)^2 = 1$$

In this case the left and right limits do *not* agree. That means that the limit as $x \to 1$ *does not exist*. So the function cannot be continuous there. (This means that the two parts of the graph don't connect.)

For (e), we have another rational function. It is continuous where defined, namely for all values of x that don't make the denominator zero. We need to factor the denominator in order to figure this out:

$$f(x) = \frac{1-x^2}{2-x-x^2} = \frac{(1-x)(1+x)}{(1-x)(2+x)}$$

So $f(x)$ is continuous for all $x \neq -2, 1$.

For (f), we have a function very similar to the function in (e). In particular, it is only different at $x = 1$. Now the function in (e) was undefined at $x = 1$ (because it made the denominator zero.) So there are two options: either the new value at $x = 1$ *fits* the old function in a continuous way (i.e. it plugs up a hole), or it doesn't.

Since $f(x)$ is the same as the function in (e), except at $x = 1$, we know that $f(x)$ is certainly continuous for all $x \neq -2, 1$. It may or may not be continuous at $x = 1$. In order to check whether or not it is continuous at $x = 1$, we need to see if the limit of $f(x)$ as $x \to 1$ is equal to $f(1) = \frac{2}{3}$. (Basically, that's just checking to see if the new point, $(1, \frac{2}{3})$ fits in with the old function.) So we compute the limit:

$$\lim_{x\to 1} \frac{1-x^2}{2-x-x^2} = \lim_{x\to 1} \frac{(1-x)(1+x)}{(1-x)(2+x)} = \lim_{x\to 1} \frac{1+x}{2+x} = \frac{2}{3}$$

We used the "factor and cancel" technique and found that the limit is, in fact $\frac{2}{3}$. Therefore, $f(x)$ is continuous at $x = 1$. So we can conclude that $f(x)$ is continuous for all $x \neq -2$.