2.6 Limits at Infinity, Horizontal Asymptotes
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1. Overview

Outline:

1. Definition of limits at infinity

2. Definition of horizontal asymptote

3. Theorem about rational powers of x

4. Infinite limits at infinity

This section is about the "long term behavior" of functions, i.e. what happens as x gets really big (positive or negative). Sometimes the function will approach a specific number as x gets big. If a function f approaches a specific number L as x gets larger and larger (positive), we say that *the limit of $f(x)$ as x approaches infinity is L* and we write:

$$\lim_{x \to \infty} f(x) = L$$

Similarly, if f approaches L as x becomes larger and larger negative,

$$\lim_{x \to -\infty} f(x) = L$$

In the first case, the graph of $f(x)$ flattens out on the right, and it approaches the line $y = L$. In the second case, the graph flattens out on the left, approaching the line $y = L$. In either case this line is called a *horizontal asymptote* of $f(x)$.

The following theorem is helpful in evaluating infinite limits:

Theorem: If r is a positive rational number,

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

Additionally, as long as x^r is *defined* for negative values of x,

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

2. Examples

Example 1: Evaluate the following limit:

$$\lim_{x \to \infty} \frac{3x + 5}{x - 4}$$

Solution:

Notice that as x gets large, the top gets large and the bottom gets large ("$\frac{\infty}{\infty}$"), so we can't immediately tell what the behavior of the function is as x gets large. We can think of this as a competition between the numerator and the denominator. Both are getting larger and larger, and the question is: Does one of them outstrip the other? Or do both the numerator and the denominator stay about even as they get larger? In order to figure this out, we need to use the following technique: divide the numerator and denominator by the highest power of x in the *denominator*. So here, we will divide top and bottom by x:

$$\lim_{x\to\infty} \frac{3x+5}{x-4} = \lim_{x\to\infty} \frac{\left(\frac{3x+5}{x}\right)}{\left(\frac{x-4}{x}\right)}$$
$$= \lim_{x\to\infty} \frac{\left(3+\frac{5}{x}\right)}{\left(1-\frac{4}{x}\right)}$$

Now we can evaluate the limit of the top and the limit of the bottom:

$$\lim_{x\to\infty} \frac{\left(3+\frac{5}{x}\right)}{\left(1-\frac{4}{x}\right)} = \frac{\lim_{x\to\infty}\left(3+\frac{5}{x}\right)}{\lim_{x\to\infty}\left(1-\frac{4}{x}\right)} = \frac{3+0}{1-0} = 3$$

So we can conclude that the numerator and the denominator stayed about even as they grew larger. (This is because the highest power of x in the numerator is the *same* as the highest power of x in the denominator.)

Notice that this also means that the function $f(x) = \frac{3x+5}{x-4}$ has a horizontal asymptote: $y = 3$.

Example 2: Evaluate the following limit:

$$\lim_{t\to-\infty} \frac{t^2+2}{t^3+t^2-1}$$

Solution:

Notice that the limit of the top is infinite and the limit of the bottom is also infinite. So we must divide the numerator and denominator by the highest power of t in the denominator, which in this case is t^3:

$$\lim_{t\to-\infty} \frac{t^2+2}{t^3+t^2-1} = \lim_{t\to-\infty} \frac{\left(\frac{t^2+2}{t^3}\right)}{\left(\frac{t^3+t^2-1}{t^3}\right)}$$
$$= \lim_{t\to-\infty} \frac{\left(\frac{1}{t}+\frac{2}{t^3}\right)}{\left(1+\frac{1}{t}-\frac{1}{t^3}\right)}$$
$$= \frac{\lim_{t\to-\infty}\left(\frac{1}{t}+\frac{2}{t^3}\right)}{\lim_{t\to-\infty}\left(1+\frac{1}{t}-\frac{1}{t^3}\right)}$$
$$= \frac{0+0}{1+0-0}$$
$$= 0$$

Since the limit is zero, we can conclude that the denominator outstripped the numerator. (This is because the highest power of t in the numerator is *smaller* than the highest power of t in the denominator.)

Notice that this means that the function $f(t) = \frac{t^2+2}{t^3+t^2-1}$ has a horizontal asymptote $y = 0$.

Example 3: Evaluate the following limit:

$$\lim_{t \to -\infty} \frac{t^3 + t^2 - 1}{t^2 + 2}$$

Solution:

Since the limit of the top is infinite and the limit of the bottom is infinite, we must divide the numerator and denominator by the highest power of t in the denominator, which in this case is t^2:

$$
\begin{aligned}
\lim_{t \to -\infty} \frac{t^3 + t^2 - 1}{t^2 + 2} &= \lim_{t \to -\infty} \frac{\left(\frac{t^3 + t^2 - 1}{t^2}\right)}{\left(\frac{t^2 + 2}{t^2}\right)} \\
&= \lim_{t \to -\infty} \frac{\left(t + 1 - \frac{1}{t^2}\right)}{\left(1 + \frac{2}{t^2}\right)} \\
&= \lim_{t \to -\infty} \frac{t + 1 - 0}{1 + 0} \\
&= \lim_{t \to -\infty} (t + 1) \\
&= -\infty
\end{aligned}
$$

So in this case the numerator outstripped the denominator. (This is because the highest power of t in the numerator is *greater* than the highest power of t in the denominator.)

Example 4: Evaluate the following limit:

$$\lim_{x \to \infty} \frac{x + 2}{\sqrt{9x^2 + 1}}$$

Solution:

This function is different from the three previous examples because it is not a *rational* function. So we have to be a little more careful. The fact that there is a square root in the denominator makes things more complicated. We have to take the square root into account when we're looking for the what power of x we should divide by. Looking at the denominator by itself we can ask, "Which term is most important, when x gets large?" In this case, the $9x^2$ is most important, because it will keep getting larger and larger, eventually overwhelming the contribution of the $+1$. So if we ignore the $+1$, we can say that when x is large, the denominator is approximately $\sqrt{9x^2}$, i.e. it is approximately $3x$. So we act as if x is the highest power of x in the demominator.

So we divide top and bottom by x and watch our algebra carefully!

$$\lim_{x \to \infty} \frac{x + 2}{\sqrt{9x^2 + 1}} = \lim_{x \to \infty} \frac{\left(\frac{x + 2}{x}\right)}{\left(\frac{\sqrt{9x^2 + 1}}{x}\right)}$$

Since x is positive, $x = \sqrt{x^2}$, and we can do the following:

$$\lim_{x\to\infty} \frac{\left(\frac{x+2}{x}\right)}{\left(\frac{\sqrt{9x^2+1}}{x}\right)} = \lim_{x\to\infty} \frac{\left(\frac{x+2}{x}\right)}{\left(\frac{\sqrt{9x^2+1}}{\sqrt{x^2}}\right)}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{x+2}{x}\right)}{\left(\sqrt{\frac{9x^2+1}{x^2}}\right)}$$

$$= \lim_{x\to\infty} \frac{\left(1 + \frac{2}{x}\right)}{\left(\sqrt{9 + \frac{1}{x^2}}\right)}$$

$$= \frac{\left(1 + 0\right)}{\left(\sqrt{9+0}\right)}$$

$$= \frac{1}{3}$$

Example 5: Find the horizontal and vertical asymptotes of $f(x) = \frac{x^2-1}{x^2-x-2}$.
Solution:

To determine whether there are horizontal asymptotes we must evaluate the limits at infinity.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-1}{x^2-x-2}$$

$$= \lim_{x\to\infty} \frac{1 - \frac{1}{x^2}}{1 - \frac{1}{x} - \frac{2}{x^2}}$$

$$= \frac{1-0}{1-0-0}$$

$$= 1$$

This means that the line $y = 1$ is a horizontal asymptote. Similarly, we can compute that:

$$\lim_{x\to-\infty} f(x) = 1$$

So $y = 1$ is the only horizontal asymptote.

To determine whether there are vertical asymptotes we check to see where the denominator is zero. If we factor the denominator, we see that:

$$x^2 - x - 2 = (x-2)(x+1)$$

so the denominator is zero when $x = -1$ and $x = 2$. So $x = -1$ and $x = 2$ are possible vertical asymptotes. If we factor the numerator as well, we can see:

$$f(x) = \frac{(x-1)(x+1)}{(x-2)(x+1)}$$

So there will be a hole at $x = -1$, because:

$$\lim_{x\to-1} \frac{(x-1)(x+1)}{(x-2)(x+1)} = \lim_{x\to-1} \frac{x-1}{x-2} = \frac{1}{3}$$

But there will be an asymptote at $x = 2$ because:

$$\lim_{x \to 2^-} \frac{(x-1)(x+1)}{(x-2)(x+1)} = \lim_{x \to 2^-} \frac{x-1}{x-2}$$

and as x gets close to 2 from the *left* the numerator gets close to 1 (positive) and the denominator gets close to zero and is negative. So,

$$\lim_{x \to 2^-} \frac{x-1}{x-2} = -\infty$$

Example 6: Find the horizontal and vertical asymptotes for $f(x) = \frac{1+2e^x}{1-e^x}$.
Solution:

To find the horizontal asymptotes we look at the limits at infinity. We know that e^x gets larger and larger as $x \to \infty$, so:

$$\lim_{x \to \infty} \frac{1+2e^x}{1-e^x} = \lim_{x \to \infty} \frac{\frac{1}{e^x}+2}{\frac{1}{e^x}-1} = \frac{0+2}{0-1} = 2$$

We know that e^x gets close to zero as $x \to -\infty$, so:

$$\lim_{x \to -\infty} \frac{1+2e^x}{1-e^x} = \frac{1+0}{1-0} = 1$$

So the lines $y = 2$ and $y = 1$ are horizontal asymptotes.

To find the vertical asymptote we notice that the denominator is zero when $x = 0$, and, so when $x \to 0^-$, the denominator is positive, close to zero and the numerator is positive, close to 3, so:

$$\lim_{x \to 0^-} \frac{1+2e^x}{1-e^x} = \infty$$

and the line $x = 0$ is a vertical asymptote.