2.7 Derivatives and Rates of Change

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1. Overview

We can define the derivative of a function $f(x)$ at a specified x-value a to be:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists. Here are three examples of the derivative occuring "in nature":

1. The slope of the tangent line to the curve $y = f(x)$, at the point $(a, f(a))$ is:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

2. The instantaneous velocity at time $t = a$ of an object whose position is specified by $s(t)$ is:

$$v = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a} = s'(a)$$

3. The instantaneous rate of change when $x = x_0$ of a quantity $Q(x)$ is:

$$r = \lim_{\Delta x \to 0} \frac{\Delta Q}{\Delta x} = \lim_{x_1 \to x_0} \frac{Q(x_1) - Q(x_0)}{x_1 - x_0} = Q'(x_0)$$

2. Examples

Example 1: Find the equation of the tangent line to the curve $y = \sqrt{2x+1}$ at the point $(4, 3)$.
Solution:

We can find an equation for a line if we know the slope of the line and a point on the line. Here, we are given a point right off the bat: $(4, 3)$, and we can find the slope by computing the derivative at $x = 4$. So the slope is:

$$\begin{aligned} m &= \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} \\ &= \lim_{x \to 4} \frac{\sqrt{2x+1} - \sqrt{2(4)+1}}{x - 4} \\ &= \lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{x - 4} \end{aligned}$$

Notice that if we plug in 4 we get zero in the numerator and the denominator. So we want to use a trick in order to get the $(x - 4)$ in the bottom to cancel out. We multiply top and bottom by the conjugate:

$$
\begin{aligned}
m &= \lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x-4} \cdot \frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3} \\
&= \lim_{x\to 4} \frac{(2x+1)-9}{(x-4)(\sqrt{2x+1}+3)} \\
&= \lim_{x\to 4} \frac{2x-8}{(x-4)(\sqrt{2x+1}+3)} \\
&= \lim_{x\to 4} \frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)} \\
&= \lim_{x\to 4} \frac{2}{(\sqrt{2x+1}+3)} \\
&= \frac{2}{(\sqrt{2(4)+1}+3)} \\
&= \frac{1}{3}
\end{aligned}
$$

So now, using the slope $m = \frac{1}{3}$ and the point $(4, 3)$ we can write the equation of the tangent line (in point-slope form):

$$y - 3 = \tfrac{1}{3}(x-4)$$

Example 2: Let $f(x) = \frac{3x+1}{x+2}$. Find $f'(a)$.
Solution:

We use the definition of the derivative at $x = a$:

$$
\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{\frac{3x+1}{x+2} - \frac{3a+1}{a+2}}{x-a} \\
&= \lim_{x\to a} \frac{(3x+1)(a+2)-(3a+1)(x+2)}{(x-a)(x+2)(a+2)} \\
&= \lim_{x\to a} \frac{(3xa+6x+a+2)-(3ax+6a+x+2)}{(x-a)(x+2)(a+2)} \\
&= \lim_{x\to a} \frac{(3xa+6x+a+2-3ax-6a-x-2)}{(x-a)(x+2)(a+2)} \\
&= \lim_{x\to a} \frac{5x-5a}{(x-a)(x+2)(a+2)} \\
&= \lim_{x\to a} \frac{5(x-a)}{(x-a)(x+2)(a+2)} \\
&= \lim_{x\to a} \frac{5}{(x+2)(a+2)} \\
&= \frac{5}{(a+2)(a+2)} \\
&= \frac{5}{(a+2)^2}
\end{aligned}
$$

Example 3: Let $f(x) = \frac{1}{\sqrt{x+2}}$. Find $f'(a)$.
Solution:

We use the definition of the derivative at $x = a$:

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{(a+h)+2}} - \frac{1}{\sqrt{a+2}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{a+2} - \sqrt{a+h+2}}{h(\sqrt{a+h+2})(\sqrt{a+2})} \\
&= \lim_{h\to 0} \frac{\sqrt{a+2} - \sqrt{a+h+2}}{h(\sqrt{a+h+2})(\sqrt{a+2})} \cdot \frac{\sqrt{a+2} + \sqrt{a+h+2}}{\sqrt{a+2} + \sqrt{a+h+2}} \\
&= \lim_{h\to 0} \frac{(a+2) - (a+h+2)}{h(\sqrt{a+h+2})(\sqrt{a+2})(\sqrt{a+2} + \sqrt{a+h+2})} \\
&= \lim_{h\to 0} \frac{-h}{h(\sqrt{a+h+2})(\sqrt{a+2})(\sqrt{a+2} + \sqrt{a+h+2})} \\
&= \lim_{h\to 0} \frac{-1}{(\sqrt{a+h+2})(\sqrt{a+2})(\sqrt{a+2} + \sqrt{a+h+2})} \\
&= \frac{-1}{(\sqrt{a+0+2})(\sqrt{a+2})(\sqrt{a+2} + \sqrt{a+0+2})} \\
&= \frac{-1}{(a+2)(2\sqrt{a+2})} \\
&= \frac{-1}{2(a+2)\sqrt{a+2}}
\end{aligned}$$