2.8 The Derivative as a Function

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1. Overview

Definition of Derivative: If we have a function $f(x)$ we can define a new function, the derivative of f to be:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists. This is the same expression we had for $f'(a)$ in the previous section. The only difference is that in the previous section we thought about $f'(a)$ as a number (which was determined by the number a), and now we're thinking of $f'(x)$ as a function depending on x.

Notation: If $y = f(x)$ the following expressions all mean the same thing:

$$f'(x) \qquad \frac{df}{dx}(x) \qquad \frac{d}{dx}f(x) \qquad y' \qquad \frac{dy}{dx}$$

Though some of these expressions look like fractions, don't treat them like fractions.

Differentiability: We say that a function is *differentiable* at $x = a$ if $f'(a)$ exists. How could $f'(a)$ fail to exist? Well, $f'(a)$ could fail to exist if:

1. f is not continuous at a (hole, asymptote, or jump)
2. f has a cusp at a
3. f has a vertical tangent at a

Note: If f is differentiable at a then it is necessarily continuous at a, but not vice versa. (Draw some pictures to convince yourself!)

Graphs: If you have the graph of a function $f(x)$ you can sketch the graph of the derivative $f'(x)$. Just think: slopes $\longrightarrow$ y-values. Here are some tips:

1. Find the "flat places" on the graph of $f(x)$, i.e. find all the x's where the slope is zero. That means that the y-value for $f'(x)$ will be zero at each of those x's.
2. Find the discontinuities, cusps, and vertical tangents on the graph of $f(x)$. Those are places where $f'(x)$ will be undefined. In particular, cusps of $f(x)$ translate into jump discontinuities of $f'(x)$, and vertical tangents of $f(x)$ translate into vertical asymptotes of $f'(x)$.
3. Intervals of increase and decrease: when $f(x)$ is increasing that means that the slopes are positive, so $f'(x)$ will have positive y-values. When $f(x)$ is decreasing that means that the slopes are negative, so $f'(x)$ will have negative y-values.
4. Estimate the slope at a few particular points. Pick some key x-values and estimate the slope of $f(x)$ at those places. Then plot those points on the graph of $f'(x)$.

2. Examples

1.) Consider $g(x) = \frac{1}{x^2}$. Use the definition of the derivative to find $g'(x)$.

We remember the definition of the derivative:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

Notice that if we plug in $h = 0$, we get "zero over zero", so our goal is to *cancel* the h in the denominator. In order to do that we will have to do some rearranging. First, multiply numerator and denominator by $x^2(x+h)^2$ to get rid of the fractions in the numerator:

$$g'(x) = \lim_{h\to 0} \frac{x^2 - (x+h)^2}{h(x^2)(x+h)^2}$$

Simplify the numerator:

$$g'(x) = \lim_{h\to 0} \frac{x^2 - (x^2 + 2hx + h^2)}{h(x^2)(x+h)^2} = \lim_{h\to 0} \frac{-2hx - h^2}{h(x^2)(x+h)^2}$$

Now we can factor an h out of the numerator and cancel it with the h in the denominator:

$$g'(x) = \lim_{h\to 0} \frac{-h(2x+h)}{h(x^2)(x+h)^2} = \lim_{h\to 0} \frac{-(2x+h)}{x^2(x+h)^2}$$

Now we have eliminated the "zero over zero" problem. Now we can just plug in $h = 0$:

$$g'(x) = \frac{-(2x+0)}{x^2(x+0)^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}$$

2.) Consider $f(x) = x|x|$. For what values is f differentiable?

This problem is a little tricky because it involves an absolute value. The absolute value function is continuous for all real numbers, but it is not differentiable at zero (because it has a corner!) The easiest way to deal with the absolute value function is to rewrite it as a piecewise function:

$$|x| = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$$

(If you are puzzling over this, try thinking of the graph.) So we can also rewrite the function $f(x)$ as a piecewise function.

$$f(x) = x|x| = \begin{cases} -x^2 & \text{if } x^2 \le 0 \\ x & \text{if } x > 0 \end{cases}$$

So on the left half plane, $f(x)$ is an upside-down parabola, and on the right half-plane $f(x)$ is a right-side-up parabola.

We have to think about where this function is differentiable. Certainly $f(x)$ is differentiable for all $x < 0$, because it is just $-x^2$ (a polynomial) on that interval. Likewise, $f(x)$ is differentiable for all $x > 0$, because it is just x^2 on that interval. So we just need to check the point $x = 0$. Remember that (by definition) $f(x)$ is continuous at 0 if and only if the limit defining $f'(0)$ (i.e. the limit of differnce quotients) exists. So we check the left and right limits of the difference quotient. The left-hand limit is:

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h^2 - 0}{h} = \lim_{h\to 0^-} -h = 0$$

The right-hand limit is:

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2 - 0}{h} = \lim_{h \to 0^-} h = 0$$

The left and right limits agree, so the limit defining $f'(0)$ exists, and $f'(x)$ is differentiable at 0. Intuitively, this means that the slope from the left matches the slope from the right, so there's a smooth transition from the left piece (the upside-down parabola) to the right piece (the right-side-up parabola.)