

3.10 Linear Approximation and Differentials

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1. Overview

Linear Approximation

We have learned how to find the tangent line to a curve at a specific point. If the function is $f(x)$, and the point is $(a, f(a))$, the equation for the tangent line (in point-slope form) is:

$$y - f(a) = f'(a) \cdot (x - a)$$

If we wanted to get y on a side by itself we write:

$$y = f'(a)(x - a) + f(a)$$

We now give the tangent line a new name: the *linear approximation* to f at a and we call it $L(x)$:

$$L(x) = f'(a)(x - a) + f(a)$$

The reason we call the tangent line the *linear approximation* is that we will use the tangent line to approximate $f(x)$ when x is near a :

$$f(x) \approx L(x) = f'(a)(x - a) + f(a) \quad \text{when } x \text{ is close to } a$$

Typically a is a “good point” (i.e. it is easy to compute $f(a)$) and x is a “bad point” (i.e. it is hard to compute $f(x)$.) For example, if we wanted to compute $\sqrt{4.001}$ we might approximate it using a linear approximation. In this case the “bad point” is 4.001 and we want to pick a point close to 4.001 whose square root is easy to compute, like, say 4. So a is the “good point” $a = 4$. The function is the square root function $f(x) = \sqrt{x}$.

Differentials

Another way to think about this is in terms of *differentials*. Let's say we have a function $f(x)$ and we can easily compute the value at a . We are wondering what happens to $f(x)$ if we move away from a just a little bit. We call the “little bit” Δx or dx . So the question is, What is $f(a + \Delta x)$? We would expect that it would not be very different from $f(a)$, since Δx is small, but we call the difference Δy :

$$\Delta y = f(a + \Delta x) - f(a)$$

We will approximate Δy in the following way. When we are looking at the point a , the function is changing at a rate of $f'(a)$. So if we multiply the rate (a.k.a. the slope of the tangent) by the interval Δx (which is the same as dx) that will give us an approximation for Δy :

$$\Delta y \approx f'(a) \cdot dx$$

We define dy to be $f'(a) \cdot dx$ and we call dx and dy differentials.

An example of this would be driving in a car. Your speed is not constant, but if you wanted to approximate how far you went over a certain time interval, you might look at your speedometer at a certain moment, multiply that speed by the time it took you to make the trip, and that would give you an approximation of how far you went. In this example, the moment you looked at the speedometer was the “good point” a , the speed at that moment is $f'(a)$, the time interval is Δx or dx , and the approximate distance traveled is $dy = f'(a) \cdot dx$.

Note: Basically this section is about using something you already know (tangent lines) to do something new (approximate values of a function.) The two methods outlined above are basically the same thing, in different terminology. In particular, the “bad point” x in the discussion of linear approximation is the same as the point $a + \Delta x$ in the discussion of differentials. The approximation $f(x) \approx L(x) = f'(a)(x - a) + f(a)$ is the same as $f(a) + \Delta y \approx f(a) + dy$, etc.

2. Examples

1.) Find the linear approximation to $f(x) = (x + 2)^5$ at $a = 0$ and use it to approximate $(2.001)^5$.

The linear approximation is:

$$L(x) = f'(a)(x - a) + f(a).$$

We start by computing the derivative of $f(x)$, because $f'(a)$ will be the slope:

$$f'(x) = 5(x + 2)^4 \quad \text{so} \quad f'(a) = f'(0) = 5(2)^4 = 5 \cdot 16 = 80$$

And we can compute $f(a) = f(0) = 2^5 = 32$. So the linear approximation is:

$$L(x) = f'(0)(x - 0) + f(0) = 80x + 32$$

Now we want to use $L(x)$ to approximate $(2.001)^5$. We need to write $(2.001)^5$ in terms of f . Notice that:

$$f(.001) = (.001 + 2)^5 = (2.001)^5$$

So we make the approximation:

$$(2.001)^5 = f(.001) \approx L(.001)$$

And we just need to compute $L(.001)$:

$$L(.001) = 80(.001) + 32 = .08 + 32 = 32.08$$

So we conclude:

$$(2.001)^5 \approx 32.08$$

2.) Suppose we measure the radius of a disk to be 24 cm, with a maximum error of .2 cm. Then we use our measurement for the radius to compute the area. Estimate the maximum error in the calculated area. What is the relative error? What is the percentage error?

We consider area as a function of radius: $A(r) = \pi r^2$. We have measured the radius to be $r = 24$ and the maximum error is $\Delta r = dr = .2$. The maximum error of the area is ΔA , which we can approximate by dA :

$$\Delta A \approx dA = A'(r) \cdot dr$$

We compute the derivative $A'(r)$:

$$A'(r) = \pi(2r) = 2\pi r$$

So when $r = 24$, $A'(r) = 48\pi$. So we compute dA :

$$dA = (48\pi)(.2) = 9.6\pi$$

So the maximum error in the area approximately $9.6\pi \text{ cm}^2$.

The relative error tells us how big the error in a quantity is *compared* to the size if the quantity. In particular:

$$\text{relative error} = \frac{\Delta A}{A}$$

So we in order to estimate the relative error we need to compute the area:

$$A = \pi \cdot (24)^2 = 576\pi$$

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{9.6\pi}{576\pi} = \frac{1}{60} \approx .0167$$

And the percentage error is 1.67%.