3.2 The Product Rule and the Quotient Rule

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1. Overview

You need to memorize the product rule and the quotient rule. And, more than that actually: you need to internalize them. The best way to do that is just by practicing until you can use them without even thinking about it.

Product Rule:

$$(fg)' = f'g + fg'$$

In words: The derivative of a product of two functions is: the derivative of the first, times the second, plus the first times the derivative of the second.

Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

In words: The derivative of a quotient of two functions is: the derivative of the top, times the bottom, minus the top times the derivative of the bottom, all over the bottom squared.

2. Examples

1.) Find the derivative of

$$g(x) = \sqrt{x}\, e^x$$

Since $g(x)$ is the product of two functions, we use the product rule:

$$g'(x) = (\sqrt{x})'(e^x) + (\sqrt{x})(e^x)'$$

Remembering that $\sqrt{x} = x^{1/2}$, we compute the derivatives:

$$g'(x) = (\tfrac{1}{2}x^{-1/2})(e^x) + (x^{1/2})(e^x)$$

And, simplifying, we get:

$$g'(x) = (\tfrac{1}{2\sqrt{x}} + \sqrt{x})\, e^x$$

2.) Find the derivative of

$$y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$$

Since y is the quotient of two functions, we use the quotient rule:

$$y' = \frac{(\sqrt{x} - 1)'(\sqrt{x} + 1) - (\sqrt{x} - 1)(\sqrt{x} + 1)'}{(\sqrt{x} + 1)^2}$$

Converting all the square roots to powers:

$$y' = \frac{(x^{1/2} - 1)'(x^{1/2} + 1) - (x^{1/2} - 1)(x^{1/2} + 1)'}{(x^{1/2} + 1)^2}$$

Now we take derivatives:

$$y' = \frac{(\frac{1}{2}x^{-1/2})(x^{1/2} + 1) - (x^{1/2} - 1)(\frac{1}{2}x^{-1/2})}{(x^{1/2} + 1)^2}$$

And we simplify the numerator:

$$
\begin{aligned}
y' &= \frac{(\frac{1}{2} + \frac{1}{2}x^{-1/2}) - (\frac{1}{2} - \frac{1}{2}x^{-1/2})}{(x^{1/2}+1)^2} \\
&= \frac{\frac{1}{2} + \frac{1}{2}x^{-1/2} - \frac{1}{2} + \frac{1}{2}x^{-1/2}}{(x^{1/2}+1)^2} \\
&= \frac{x^{-1/2}}{(x^{1/2}+1)^2}
\end{aligned}
$$

Rewriting in terms of the square roots:

$$y' = \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}$$