3.4 The Chain Rule

Math 1271, TA: Amy DeCelles

1. Overview

You need to memorize and internalize the chain rule. Again, the best way to do this is just by practicing until you can do it without thinking about it.

Chain Rule:

$$(f(g(x))' = f'(g(x)) \cdot g'(x)$$

In words: Take the derivative of the outer function, plug in the inner function, and multiply by the derivative of the inner function.

Another way to write the chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Derivatives of Exponentials: In this section, we used the chain rule to figure out what the derivatives of exponential functions (with base $a \neq e$) are:

$$(a^x)' = (\ln a) \cdot a^x$$

Note: Do not use the power rule to take the derivative of a^x! The power rule is used when the base is x and the exponent is a number. The exponential rule is used when the base is a number and the exponent is x!

2. Examples

1.) Find the derivative of:

$$f(x) = (1 + x^4)^{2/3}$$

The first step to a chain rule problem is to recognize the inner function and the outer function. In this case, the outer function is $u^{2/3}$ and the inner function is $1 + x^4$. (You plug the inner function into the outer function to get $f(x)$.)

To compute the derivative of the outer function, we just use the power rule:

$$(u^{2/3})' = \tfrac{2}{3}u^{1-2/3} = \tfrac{2}{3}u^{-1/3}$$

To compute the derivative of the inner function, we also use the power rule:

$$(1 + x^4)' = 0 + 4x^3 = 4x^3$$

Now using the chain rule we can find the derivative of f:

$$f'(x) = \tfrac{2}{3}(1 + x^4)^{-1/3} \cdot (4x^3) = \tfrac{8}{3}x^3(1 + x^4)^{-1/3}$$

2.) Find the derivative of:

$$f(x) = \sqrt[3]{1 + \tan x}$$

The outer function is $\sqrt[3]{u} = u^{1/3}$ and the inner function is $1 + \tan x$. The derivative of the outer function is:

$$(u^{1/3})' = \tfrac{1}{3}u^{1-1/3} = \tfrac{1}{3}u^{-2/3}$$

And the derivative of the inner function is:

$$(1 + \tan x)' = 0 + \sec^2 x = \sec^2 x$$

So using the chain rule:

$$f'(x) = \tfrac{1}{3}(1 + \tan x)^{-2/3} \cdot (\sec^2 x)$$

3.) Find the derivative of:

$$y = 5 + \cos^3 x$$

Well, the derivative of 5 is zero, so:

$$y' = 0 + (\cos^3 x)' = (\cos^3 x)'$$

Notice that $\cos^3 x = (\cos x)^3$. We have to use the chain rule here, with outer function u^3 and inner function $\cos x$. We get:

$$y' = 3(\cos x)^2 \cdot (-\sin x) = -3\cos^2 x \sin x$$

4.) Find the derivative of:

$$y = e^{-5x} \cos 3x$$

Notice that this is a product of two functions e^{-5x} and $\cos 3x$, so we use the product rule first:

$$y' = (e^{-5x})'(\cos 3x) + (e^{-5x})(\cos 3x)'$$

To take the derivative of e^{-5x} we have to use the chain rule, because $-5x$ is plugged into e^u. So using the chain rule:

$$(e^{-5x})' = (e^{-5x}) \cdot (-5) = -5e^{-5x}$$

To take the derivative of $\cos 3x$, we have to use the chain rule, because $3x$ is plugged into $\cos u$. So we get:

$$(\cos 3x)' = (-\sin 3x) \cdot (3) = -3\sin 3x$$

Putting it all back together, we get:

$$\begin{aligned} y' &= (-5e^{-5x})(\cos 3x) + (e^{-5x})(-3\sin 3x) \\ &= -5e^{-5x}\cos 3x - 3e^{-5x}\sin 3x \\ &= -(5\cos 3x + 3\sin 3x)e^{-5x} \end{aligned}$$

5.) Find the derivative of:

$$y = \sin^2 5x$$

Here we remember that $\sin^2 5x = (\sin 5x)^2$. We will have to use the chain rule twice. For the first time, the outer function is u^2 and the inner function is $\sin 5x$. So we get:

$$y' = 2(\sin 5x) \cdot (\sin 5x)'$$

To take the derivative of $\sin 5x$ we need to use the chain rule again. This time the outer function is $\sin u$ and the inner function is $5x$. So we get:

$$(\sin 5x)' = (\cos 5x) \cdot (5) = 5 \cos 5x$$

Putting it all back together, we get:

$$y' = 2(\sin 5x) \cdot (5 \cos 5x) = 10 \sin 5x \cos 5x$$