## 3.5 Implicit Differentiation

Math 1271, TA: Amy DeCelles

## 1. Overview

So far we have looked at finding $\frac{dy}{dx}$ when $y$ is defined explicitly by a function of $x$, i.e. $y = f(x)$. Now we will look at finding $\frac{dy}{dx}$ when the relationship between $x$ and $y$ might not be so simple. For example, we might have an equation with $x$'s and $y$'s on both sides, and it might not be possible to get the $y$ on a side by itself. This means that $y$ is defined *implicitly*. The method of finding $\frac{dy}{dx}$ in such a case is called *implicit differentiation*.

**Implicit Differentiation:**

1. Take $\frac{d}{dx}$ of both sides.
2. Take derivatives remembering that $y$ is a function of $x$. Use the product rule, quotient rule, chain rule where appropriate. For example:

   Product Rule: $\frac{d}{dx}(\sin x)(y) = (\cos x)(y) + (\sin x)(\frac{dy}{dx})$

   Quotient Rule: $\frac{d}{dx}(\frac{x^2}{y}) = \frac{2xy - x^2(\frac{dy}{dx})}{y^2}$

   Chain Rule: $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$
3. Get $\frac{dy}{dx}$ on a side by itself.

**Derivatives of Inverse Trig Functions:**
You need to know the derivatives of the inverse trig functions. (Memorize them!)

$$\begin{aligned}
(\sin^{-1} x)' &= \frac{1}{\sqrt{1-x^2}} \\
(\cos^{-1} x)' &= -\frac{1}{\sqrt{1-x^2}} \\
(\tan^{-1} x)' &= \frac{1}{1+x^2} \\
(\sec^{-1} x)' &= \frac{1}{x\sqrt{x^2-1}} \\
(\cot^{-1} x)' &= -\frac{1}{1+x^2} \\
(\csc^{-1} x)' &= -\frac{1}{x\sqrt{x^2-1}}
\end{aligned}$$

**Note:** Another notation for the inverse trig functions is: $\arcsin x = \sin^{-1} x$, $\arccos x = \cos^{-1} x$, etc. You should be familiar with both ways of writing inverse trig functions.

## 2. Examples

1.) Find $\frac{dy}{dx}$ when

$$4x^2 + 9y^2 = 36$$

We start by taking $\frac{d}{dx}$ of both sides:

$$\tfrac{d}{dx}(4x^2 + 9y^2) = \tfrac{d}{dx}(36)$$

Certainly the right hand side is zero, because the derivative of a constant number is zero. On the left hand side we get:

$$\tfrac{d}{dx}(4x^2 + 9y^2) = \tfrac{d}{dx}(4x^2) + \tfrac{d}{dx}(9y^2) = 8x + 9(2y\tfrac{dy}{dx})$$

(using the chain rule on $y^2$ since the function $y$ is plugged into the function $u^2$!)

Putting the right hand side and the left hand side back together, we get:

$$8x + 18y\tfrac{dy}{dx} = 0$$

We solve to get $\frac{dy}{dx}$ on a side by itself:

$$\begin{aligned}
18y\tfrac{dy}{dx} &= -8x \\
\tfrac{dy}{dx} &= -\frac{8x}{18y} \\
\tfrac{dy}{dx} &= -\frac{4x}{9y}
\end{aligned}$$

And we are done! (Notice that the answer has $x$ and $y$ in it. This is normal.

2.) Find $\frac{dy}{dx}$ when

$$\sqrt{x} + \sqrt{y} = 4$$

We start by taking $\frac{d}{dx}$ of both sides:

$$\tfrac{d}{dx}(\sqrt{x} + \sqrt{y}) = \tfrac{d}{dx}(4)$$

Certainly the right hand side is zero. On the left hand side we get:

$$\tfrac{d}{dx}(\sqrt{x} + \sqrt{y}) = \tfrac{d}{dx}(x^{1/2}) + \tfrac{d}{dx}(y^{1/2}) = \tfrac{1}{2}x^{-1/2} + \tfrac{1}{2}y^{-1/2} \cdot \tfrac{dy}{dx}$$

(using the chain rule on $y^{1/2}$.)

Putting the right hand side and the left hand side back together, we get:

$$\tfrac{1}{2}x^{-1/2} + \tfrac{1}{2}y^{-1/2} \cdot \tfrac{dy}{dx} = 0$$

Rewriting:

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

Now we solve to get $\frac{dy}{dx}$ on a side by itself:

$$\begin{aligned}
\frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} &= -\frac{1}{2\sqrt{x}} \\
\frac{dy}{dx} &= -\frac{2\sqrt{y}}{2\sqrt{x}} \\
\frac{dy}{dx} &= -\frac{\sqrt{y}}{\sqrt{x}} \\
\frac{dy}{dx} &= -\sqrt{\frac{y}{x}}
\end{aligned}$$

3.) Find $y'$ when

$$y^5 + x^2y^3 = 1 + ye^{x^2}$$

We start by taking the derivative of both sides with respect to $x$:

$$5y^4y' + (2x \cdot y^3 + x^2 \cdot 3y^2y') = 0 + (y' \cdot e^{x^2} + y \cdot e^{x^2} \cdot 2x)$$

(We used the chain rule on $y^5$, the product rule and the chain rule on $x^2y^3$, and the product rule and the chain rule on $ye^{x^2}$.)

Simplifying:

$$5y^4y' + 2xy^3 + 3x^2y^2y' = y'e^{x^2} + 2xye^{x^2}$$

We aim to get $y'$ on a side by itself, so we move all terms containing a $y'$ to the left hand side, and all terms without a $y'$ to the right hand side:

$$5y^4y' + 3x^2y^2y' - y'e^{x^2} = 2xye^{x^2} - 2xy^3$$

We factor out a $y'$ on the left hand side:

$$(5y^4 + 3x^2y^2 - e^{x^2})y' = 2xye^{x^2} - 2xy^3$$

Now divide both sides by $(5y^4 + 3x^2y^2 - e^{x^2})$ to get $y'$ by itself:

$$y' = \frac{2xye^{x^2} - 2xy^3}{5y^4 + 3x^2y^2 - e^{x^2}}$$

4.) Find $y'$:

$$y = \sqrt{\arctan x}$$

Use the chain rule with the outer function being $\sqrt{u} = y^{1/2}$ and the inner function being $\arctan x$:

$$y' = \tfrac{1}{2}(\arctan x)^{-1/2} \cdot \frac{1}{1+x^2} = \frac{1}{2\sqrt{\arctan x}} \cdot \frac{1}{1+x^2} = \frac{1}{2\sqrt{\arctan x}(1+x^2)}$$