3.8 Exponential Growth and Decay

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1. Overview

This section discusses several natural phenomena (population growth, radioactive decay, Newton's law of cooling, continuously compounded interest) from a mathematical perspective. In each of these examples, there is a quantity that is changing, and in particular, the rate at which it is changing is proportional to the quantity. Putting this into symbols, if y is a quantity changing with respect to x, then the fact that the rate at which y is changing is proportional to y means:

$$\frac{dy}{dx} = k \cdot y \quad \text{for some proportionality constant } k$$

There is a theorem that says that any quantity satisfying this relationship is an exponential. In particular:

$$y(x) = y_0 \cdot e^{kx} \quad \text{where } y_0 = y(0)$$

Growth vs Decay

Population growth is an example of exponential growth. The amount of bacteria in a petrie dish will increase at a rate proportional the the amount of bacteria present. So the more and more bacteria there are, the faster the population will grow. More specifically, given a set time interval, say one hour, the amount of bacteria will multiply by the same factor. So, say there are 100 bacteria at 2:00 and 200 bacteria at 3:00. Then there will be 400 bacteria at 4:00, 800 bacteria at 5:00, etc. The amount doubles every hour. The equation for exponential growth in this example is:

$$P(t) = 100 \cdot 2^t$$

In general, exponential growth will be:

$$P(t) = P_0 \cdot a^t \quad \text{where } a > 1 \text{ is the multiplication factor per unit time}$$

or equivalently:

$$P(t) = P_0 \cdot e^{kt} \quad \text{where } k = \ln a > 0$$

Radioactive decay is an example of exponential decay. The amount of Carbon 14 in a sample *decreases* at a rate proportional to the amount present. So the less and less Carbon 14 there is the slower and slower it goes away. Again, more specifically, given a set time interval, say 100 years, the amount of Carbon 14 will decrease by the same factor. So if a sample starts with 900 miligrams of Carbon 14, and 100 years later there is 300 miligrams, then we can say that after 200 years there will be 100 miligrams, after 300 years there will be 33 miligrams, etc. The amount is divided by three every 100 years. The equation for exponential decay in this example is:

$$Q(t) = 900 \cdot (\tfrac{1}{3})^t \quad \text{where } t \text{ is measured in hundreds of years}$$

If we want t to represent the number of years, we have to divide it by 100 in the equation:

$$Q(t) = 900 \cdot (\tfrac{1}{3})^{t/100} \quad \text{where } t \text{ is measured in years}$$

In general exponential decay will be:

$$Q(t) = Q_0 \cdot a^t \quad \text{where } a < 1 \text{ is the multiplication factor per unit time}$$

or equivalently:

$$Q(t) = Q_0 \cdot e^{kt} \quad \text{where } k = \ln a < 0$$

NB: I made these examples up. I am not a biologist or a physicist or a geologist ... so the particular numbers in the examples may not be realistic.

Newton's Law of Cooling

Suppose you have a cup of hot cocoa (or some other liquid). Assuming you're in a room that is cooler than your hot cocoa, there is a temperature difference between the hot cocoa and the room temperature. Newton's law of cooling says that that temperature difference will decrease at a rate proportional to the current temperature difference. So if we assume the room temperature doesn't change, this means as your hot cocoa gets cooler and cooler, it cools down slower and slower.

If we call the temperature difference D, then what we have just said is that:

$$\frac{dD}{dt} = k \cdot D \quad \text{for some number } k$$

So by the theorem mentioned earlier:

$$D(t) = D_0 \cdot e^{kt}$$

We want to know the temperature of the hot cocoa at time t, not just the difference in temperature. If we let T_r stand for the room temperature and T stand for the temperature of the hot cocoa, then $D = T - T_r$, so we get:

$$T(t) - T_r(t) = (T(0) - T_r(0)) \cdot e^{kt}$$

If we assume that the room temperature does not change, $T_r(0) = T_r(t) = T_r$, so we get:

$$T(t) - T_r = (T_0 - T_r) \cdot e^{kt}$$

So the temperature of the hot cocoa at time t is:

$$T(t) = (T_0 - T_r) \cdot e^{kt} + T_r$$