4.1 Maximum and Minimum Values

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1. Overview

Outline:

1. Definition of absolute and local maximum and minimum values of a function
2. Theorem: Local max/min values will always occur at a *critical number*, i.e. a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.
3. Theorem: A continuous function will always have an absolute max value and an absolute min value on a closed interval.
4. Finding abs max/min values (for a continuous function on a closed interval)
   - Either the max/min value occurs at an endpoint, or it occurs somewhere in the middle.
   - If it occurs in the middle, it has to be a local max/min value as well, and we know that local max/min values always occur at critical numbers.
   - So the abs max/min will be either at an endpoint or a critical number.
   - You just have to check the value of the function at the endpoints and the critical numbers and see what value is the greatest and which the least; those are your absolute max and min values.

Note: We will learn how to find local max/min values in section 4.3.

You should be able to look at a graph of a function and determine:

1. the critical numbers
2. the local min/max values
3. the absolute min/max values

2. Examples

1.) Find the critical numbers of

$$g(x) = \sqrt{1 - x^2}$$

Remember that the critical numbers of g are numbers in the domain of g where $g'(x) = 0$ or $g'(x)$ DNE. So we start by taking the derivative (have to use the chain rule):

$$g'(x) = \tfrac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{1 - x^2}}$$

When is $g'(x) = 0$? Just when $x = 0$ (because the numerator will be zero, and the denominator will be nonzero.)

When does $g'(x)$ DNE? Well, $g'(x)$ DNE if the denominator is zero, i.e. if $x = \pm 1$, and $g'(x)$ also DNE if the inside of the square root is negative, i.e. if:

$$\begin{aligned} 1 - x^2 &< 0 \\ 1 &< x^2 \\ \sqrt{1} &< |x| \\ 1 &< |x| \\ x > 1 \quad &\text{or} \quad x < -1 \end{aligned}$$

So $g'(x)$ DNE if $x \geq 1$ or $x \leq -1$.

To sum up: $g'(x) = 0$ for $x = 0$ and $g'(x)$ DNE for $x \geq 1$ or $x \leq -1$. Now we just need to check which of these numbers are in the domain of g. The domain of g is going to be all real numbers, except those numbers that make the inside of the square root negative. We've already figured out that the square root is negative when $x > 1$ or $x < -1$, so the domain of g is $[-1, 1]$. So the critical numbers are $-1, 0$, and 1.

2.) Find the absolute max/min values of $f(x) = x^3 - 3x + 1$ on the interval $[0, 3]$.

We know that the absolute max/min values of $f(x)$ will occur either at an endpoint or a critical number. So we start by finding the critical numbers. We need to take the derivative:

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

Where is $f'(x) = 0$? If we factor $f'(x)$,

$$f'(x) = 3(x - 1)(x + 1)$$

we can see that $f'(x) = 0$ when $x = \pm 1$. Where does $f'(x)$ DNE? Nowhere. Since the numbers ± 1 are in the domain of f, ± 1 are the critical numbers of $f(x)$.

So we need to check the value of $f(x)$ at each endpoint of $[0, 3]$ and each critical number in $[0, 3]$. Since -1 is not in the interval, we don't have to check $f(x)$ there.

$$\begin{aligned} f(0) &= 0 - 0 + 1 = 1 \\ f(1) &= 1 - 3(1) + 1 = -1 \\ f(3) &= 3^3 - 3(3) + 1 = 27 - 9 + 1 = 19 \end{aligned}$$

So the absolute max value is 19 and the absolute min value is -1.

3.) Find the absolute max/min values of $f(x) = \frac{x^2-4}{x^2+4}$ on the interval $[-4, 4]$.

We know that the absolute max/min values of $f(x)$ will occur either at an endpoint or a critical number. So we start by finding the critical numbers. We take the derivative using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(2x)(x^2 + 4) - (x^2 - 4)(2x)}{(x^2 + 4)^2} \\ &= \frac{2x((x^2 + 4) - (x^2 - 4))}{(x^2 + 4)^2} \\ &= \frac{2x(x^2 + 4 - x^2 + 4)}{(x^2 + 4)^2} \\ &= \frac{16x}{(x^2 + 4)^2} \end{aligned}$$

We can see that $f'(x) = 0$ just when $x = 0$, and that there are no values of x where $f'(x)$ DNE. Since $x = 0$ is in the domain of f, it is a critical number.

So we need to check the value of $f(x)$ at each endpoint of $[-4,4]$ and each critical number in $[-4,4]$:

$$\begin{aligned} f(-4) &= \tfrac{0}{8} = 0 \\ f(0) &= \tfrac{-4}{4} = -1 \\ f(4) &= \tfrac{0}{8} = 0 \end{aligned}$$

So the absolute max value is 0 and the absolute min value is -1.

Note: There are two places where the absolute max value is attained, $x=-4$ and $x=4$. This is not a problem. The value $y=0$ is still the highest value that $f(x)$ achieves on the interval.