

## 4.2 The Mean Value Theorem

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### 1. Overview

This section contains two big theorems: Rolle's Theorem and the Mean Value Theorem (MVT). You must:

1. know the statement of each theorem (word for word)
2. understand the meaning of the theorems (i.e. be able to explain it in your own words)
3. recognize when the theorem would be helpful for proving something

**Rolle's Theorem:** Suppose  $f$  is a function satisfying the following three conditions:

1.  $f$  is continuous on a closed interval  $[a, b]$
2.  $f$  is differentiable on the open interval  $(a, b)$
3.  $f(a) = f(b)$

Then we can conclude: There is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .

**Note:** One way we use Rolle's Theorem is in showing that a certain equation does not have more than one root. Say the equation is  $f(x) = 0$ , where  $f$  is continuous and differentiable everywhere (to keep things simple for the moment). Then we would say, *if* the equation had more than one root, we could name two of the roots  $a$  and  $b$ , and then  $f$  would satisfy the three conditions of Rolle's Theorem, and so we could conclude that there is a number  $c$  between  $a$  and  $b$  such that  $f'(c) = 0$ . In other words, we're saying that if the equation  $f(x) = 0$  has more than one root, then the derivative  $f'(x)$  must be zero somewhere. That is something we can easily check. If we can prove that  $f'(x)$  is always positive (or always negative), then  $f'(x)$  can't be zero, so the equation can't have more than one root.

**Mean Value Theorem:** Suppose  $f$  is a function satisfying the following two conditions:

1.  $f$  is continuous on a closed interval  $[a, b]$
2.  $f$  is differentiable on the open interval  $(a, b)$

Then we can conclude: There is a number  $c$  in  $(a, b)$  such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

To put a physical interpretation on this, think of driving a car. I could find my average speed by dividing the distance I traveled by the time it took me. The mean (meaning average!) value theorem says that there was (at least one) moment during my trip when my speed was exactly equal to my average speed.

There are a number of consequences of the MVT that are helpful to know:

1. Upper Bound: If the derivative is bounded on  $[a, b]$ , i.e.  $f'(x) \leq M$  on  $(a, b)$ , then  $f(x) \leq M(x - a) + f(a)$  for all  $x$  in  $[a, b]$ .
2. If the derivative is zero on  $(a, b)$ , then the original function is a constant on  $(a, b)$ .
3. If the derivatives of two functions are the same on  $(a, b)$ , then the two functions differ by a constant on  $(a, b)$ .

## 2. Examples

- 1.) Verify that the function  $f(x) = x^3 + x - 1$  satisfies the hypotheses of the Mean Value Theorem on the interval  $[0, 2]$ , and find all numbers  $c$  that satisfy the conclusion of the Mean Value Theorem.

This problem is basically asking you to check for yourself that the MVT is true for this particular function on this particular interval. You do not have to *use* the MVT to prove anything.

We have to check that  $f(x)$  is continuous on the closed interval  $[0, 2]$  and differentiable on the open interval  $(0, 2)$ . Since  $f(x)$  is a polynomial it is continuous everywhere, so it is certainly continuous on  $[0, 2]$ . The derivative  $f'(x) = 3x^2 + 1$  exists for all  $x$ , so that means that  $f$  is differentiable for all  $x$ , and in particular  $f$  is differentiable on  $(0, 2)$ . So the hypotheses of the MVT are satisfied.

Now we have to find all numbers  $c$  that satisfy the conclusion of the MVT, i.e. all  $c$  in  $[0, 2]$  such that:

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

We compute the LHS:

$$\frac{f(2) - f(0)}{2 - 0} = \frac{(8 + 2 - 1) - (0 + 0 - 1)}{2} = \frac{8 + 2}{2} = 5$$

So we have to find all  $c$  between 0 and 2 such that  $f'(c) = 5$ . So we just set the derivative equal to 5 and solve for  $x$ :

$$\begin{aligned} 3x^2 + 1 &= 5 \\ 3x^2 - 4 &= 0 \end{aligned}$$

Using the quadratic formula:

$$x = \frac{0 \pm \sqrt{0 - 4(3)(-4)}}{2(3)} = \frac{\pm 4\sqrt{3}}{6} = \pm \frac{2\sqrt{3}}{3}$$

Of these two, only  $\frac{2\sqrt{3}}{3}$  is in the interval  $[0, 2]$ . (Clearly it is positive. You can tell that it is less than 2 because  $\sqrt{3} < 3$ .) So there is only one number  $c$  satisfying the conclusion of the MVT and it is  $c = \frac{2\sqrt{3}}{3}$ .

Just to reiterate, we did not *use* the MVT in this problem, we just checked that it was true. If a problem asks you to use the MVT you will typically *not* be able to *find*  $c$ . The power of the MVT is that it is able to tell us that such a  $c$  exists even if we have no way of computing it precisely.

2.) Show that the equation  $3x + 2 \cos x + 5 = 0$  has exactly one real root.

Showing that the equation has *exactly one* real root means that we have to show two things:

1. The equation has a real root.
2. The equation does not have more than one real root.

We will use the IVT (from way back in Chapter 2!) to show the first part, and we will use Rolle's theorem to show the second part.

We let  $f(x)$  equal the LHS of the equation:

$$f(x) = 3x + 2 \cos x + 5$$

So then showing that there is a number  $c$  such that  $f(c) = 0$  will show that the equation has a root. In order to show this we need to show that  $f(x)$  is negative somewhere and  $f(x)$  is positive somewhere. Since  $x = 0$  is easy to plug in, we try that:

$$f(0) = 0 + 2 + 5 > 0$$

So  $f(x)$  is positive when  $x = 0$ . To find a place where  $f(x)$  is negative we just pick a large negative number, say  $-100$ . Then:

$$f(-100) = -300 + 2 \cos(-100) + 5$$

Since  $\cos(-100)$  has to be between  $-1$  and  $1$ ,  $f(-100)$  will certainly be negative. So we have an interval  $[-100, 0]$ . We need to check the hypotheses of the IVT:

1.  $f(x)$  is certainly continuous on  $[-100, 0]$  because it is the sum of continuous functions. ✓
2.  $f(-100) \neq f(0)$  ✓
3. If we let  $N = 0$ ,  $f(-100) < N < f(0)$  ✓

So we can conclude, by the IVT, that there is a number  $c$  between  $-100$  and  $0$  such that  $f(c) = N = 0$ . So the equation has a root.

Now we use Rolle's theorem to show that the equation cannot have more than one root. We will make an argument like this: *if* the equation had more than one root, then (blah) ... but (blah) is not true, so the equation cannot have more than one root.

Ok, so we suppose for the moment that the equation has more than one root. Name two of the roots  $a$  and  $b$ . Then because of the way we have it set up,  $f(a) = 0 = f(b)$  (which is the set-up for Rolle's theorem!) Since  $f(x)$  is continuous everywhere it is certainly continuous on  $[a, b]$ . And the derivative  $f'(x) = 3 - 2 \sin x$  exists for all  $x$ , so that means  $f(x)$  is differentiable everywhere, and in particular on the open interval  $(a, b)$ . So, by Rolle's theorem, there is a number  $c$  such that  $f'(c) = 0$ .

So far we have shown that *if* the equation has more than one root, then  $f'(c) = 0$  for some number  $c$ . But we can check whether or not that is true. As we have already computed,  $f'(x) = 3 - 2 \sin x$ , but this is never equal to zero, because  $\sin x$  is always

between  $-1$  and  $1$ ! So there is no such  $c$ . So that means that the equation cannot have more than one root.

Now, we have shown (using the IVT) that the equation has a root and (by Rolle's theorem) that the equation cannot have more than one root. So the equation has *exactly one* real root.