4.3 How Derivatives Effect the Shape of a Graph

Math 1271, TA: Amy DeCelles

1. Overview

Increasing and Decreasing

The *first* derivative gives increasing/decreasing information about the original function:

$f'(x)$ *positive* $\longrightarrow$ slope of the tangent is positive $\longrightarrow$ $f(x)$ is *increasing*

$f'(x)$ *negative* $\longrightarrow$ slope of the tangent is negative $\longrightarrow$ $f(x)$ is *decreasing*

The only places where f can switch from increasing to decreasing are when $f'(x) = 0$ or $f'(x)$ DNE.

Note: Watch out, these numbers are not necessarily critical numbers, because critical numbers *have to be in the domain of* $f(x)$! For example $f(x) = \frac{1}{x^2}$ switches from increasing to decreasing at $x = 0$, but $x = 0$ is *not* a critical number because it is *not* in the domain of f.

Local Maxima and Minima

Remember from 4.1 that critical numbers are the only possibilities where local max/min may occur. (A local max/min surely must occur at a place in the domain where f switches from increasing to decreasing or decreasing to increasing.) So we find the places where local max/min occur by checking each critical number c:

$f'(x)$ negative to the left of c, $f'(x)$ positive to the right of c $\longrightarrow$ local min at c

$f'(x)$ positive to the left of c, $f'(x)$ negative to the right of c $\longrightarrow$ local max at c

If the sign of the derivative is the *same* on both sides of c, then there is neither a local min nor a local max at c. This way of checking the critical numbers is called the *first derivative test.*

Concavity

The *second* derivative gives concavity information about the original function:

1. $f''(x)$ *positive* $\longrightarrow$ $f(x)$ concave *up*
2. $f''(x)$ *negative* $\longrightarrow$ $f(x)$ concave *down*

The only places where f can switch concavity are when $f''(x) = 0$ or $f''(x)$ DNE.

Inflection Points

A point (x, y) on the graph of $f(x)$ is called an *inflection point* if f switches concavity at x. (Note that an inflection point is a *point* with an x-value and a y-value.)

Note: Just because a function switches concavity at x, that does not mean it will have an inflection point there. For example, $f(x) = \frac{1}{x}$, $f(x)$ switches concavity at $x = 0$, but $f(x)$ is undefined at $x = 0$, so there is no inflection point there.

Local Maxima and Minima Revisited

Another way to check a critical number to see if a local max/min occurs there, is by checking concavity instead of increasing/decreasing. This is called the *second derivative test.* You can do this as long as $f''(c)$ exists.

$f''(c)$ positive $\longrightarrow$ f concave *up* at c $\longrightarrow$ local *min* at c

$f''(c)$ negative $\longrightarrow$ f concave *down* at c $\longrightarrow$ local *max* at c

2. Examples

1.) Consider $f(x) = 5 - 3x^2 + x^3$. Find the intervals of increase and decrease, the local maximum and minimum values, the intervals of concavity, and the inflection points.

We will use the first derivative to find the intervals of increase and decrease, so we compute:

$$f'(x) = -6x + 3x^2 = 3x(x-2)$$

We will use the second derivative to find the intervals of concavity, so we compute:

$$f''(x) = -6 + 6x = 6(x-1)$$

To find the intervals of increase and decrease, we find all the places where $f'(x) = 0$ or $f'(x)$ DNE (because those are the places where the derivative could switch from positive to negative or vice versa.) In this example $f'(x)$ exists for all x, so we just have to look at the places where $f'(x) = 0$, i.e. $x = 0$ and $x = 2$. Now these x-values divide up the real line into three open intervals, $(-\infty, 0)$, $(0, 2)$, and $(2, +\infty)$. We check the sign of $f'(x)$ on each interval:

On $(-\infty, 0)$,

$$f'(x) = 3x(x-2) = (-)(-) = (+)$$

On $(0, 2)$,

$$f'(x) = 3x(x-2) = (+)(-) = (-)$$

On $(2, +\infty)$,

$$f'(x) = 3x(x-2) = (+)(+) = (+)$$

So $f(x)$ is increasing on $(-\infty, 0)$ and $(2, +\infty)$ and decreasing on $(0, 2)$.

Since $x = 0$ and $x = 2$ are both in the domain of $f(x)$, they are both critical numbers, and by the increasing/decreasing information we can tell that there must be a local max at $x = 0$ and a local min at $x = 2$. The local max value is $f(0) = 5 - 0 + 0 = 5$ and the local min value is $f(2) = 5 - 3(4) + 8 = 1$.

To find the intervals of concavity, we find all the places where $f''(x) = 0$ or $f''(x)$ DNE (because those are the places where the second derivative could switch from positive to negative or vice versa.) In this example $f''(x)$ exists for all x, so we just have to look at the places where $f''(x) = 0$, i.e. $x = 1$. So this divides up the real line into two open intervals $(-\infty, 1)$ and $(1, +\infty)$. We check the sign of $f''(x)$ on each interval:

On $(-\infty, 1)$,

$$f''(x) = 6(x-1) = (-)$$

On $(1, \infty)$,

$$f''(x) = 6(x-1) = (+)$$

So $f(x)$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

Since $x = 1$ is in the domain of $f(x)$ and the concavity switches at $x = 0$, there is an inflection point there. To find the y-value we just plug $x = 1$ in to the original function: $f(1) = 5 - 3 + 1 = 3$. So the point $(1, 3)$ is an inflection point.