4.4 Indeterminate Forms and L'Hospital's Rule

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1. Overview

L'Hospital's Rule

This section gives us a way to evaluate limits of functions that look like "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$". The trick is to use L'Hospital's rule (pronounced low-pea-tahl, not la-hospital), which says that you can take the derivative of the top and the derivative of the bottom and *then* take the limit of *that*. More precisely:

Suppose $F(x)$ is a quotient $F(x) = \frac{f(x)}{g(x)}$ (where f and g are differentiable and $g'(x) \neq 0$ near a, except possibly at a.)

Suppose that we are in one of the following two cases:

Case 1: ($F(x)$ is like "$\frac{0}{0}$" as $x \to a$)

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

Case 2: ($F(x)$ is like "$\frac{\infty}{\infty}$" as $x \to a$)

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

Then we can do the following:

$$\lim_{x\to a} F(x) = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

as long as the limit on the RHS exists (or is $\pm\infty$.)

Note: This also works for right/left hand limits ($\lim_{x\to a^\pm}$) and limits at infinity $\lim_{x\to\pm\infty}$).

Note: Do *not* confuse this with the quotient rule! We are *not* taking the derivative of $F(x)$ when we apply L'Hospital's rule. We do take the derivative of the top and the derivative of the bottom in order to evaluate the limit.

Variations

1. Intederminate Products "$0 \cdot \infty$": Write $f(x)g(x)$ as a quotient:

$$f(x)g(x) = \frac{f(x)}{\frac{1}{g(x)}} \quad \text{or} \quad \frac{g(x)}{\frac{1}{f(x)}}$$

   Then use L'Hospital's rule.

2. Inteterminate Differences "$\infty - \infty$": Rewrite as quotient using a common denominator, rationalization, or factoring out a common factor. Then use L'Hospital's rule.

3. Indeterminate Powers "0^0", "∞^0", or "1^∞": Can take ln of both sides or rewrite as:

$$[f(x)]^{g(x)} = e^{g(x)\cdot\ln(f(x))}$$

2. Examples

1.) Evaluate the following limit:

$$\lim_{x\to -2} \frac{x+2}{x^2+3x+2}$$

We can do this problem two ways, one using the methods of Chapter 2, and the other using L'Hospital's rule. We'll do both ways.

Always we start by plugging in the number:

$$\frac{-2+2}{4+(-6)+2} = \text{“}\frac{0}{0}\text{”}$$

So we can try to factor and cancel:

$$\lim_{x\to -2} \frac{x+2}{x^2+3x+2} = \lim_{x\to -2} \frac{x+2}{(x+2)(x+1)} = \lim_{x\to -2} \frac{1}{x+1} = \frac{1}{-2+1} = -1$$

Now we compute the limit again using L'Hospital's rule.

$$\lim_{x\to -2} \frac{x+2}{x^2+3x+2} \overset{H}{=} \lim_{x\to -2} \frac{(x+2)'}{(x^2+3x+2)'} = \lim_{x\to -2} \frac{1}{2x+3} = \frac{1}{2(-2)+3} = -1$$

2.) Evaluate the following limit:

$$\lim_{x\to 0} \frac{x+\tan x}{\sin x}$$

If we plug in 0, we get “$\frac{0}{0}$”, so we can use L'Hospital's rule:

$$\lim_{x\to 0} \frac{x+\tan x}{\sin x} \overset{H}{=} \lim_{x\to 0} \frac{(x+\tan x)'}{(\sin x)'} = \lim_{x\to 0} \frac{1+\sec^2 x}{\cos x} = \frac{1+1}{1} = 2$$

3.) Evaluate the following limit:

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x}$$

As $x \to \infty$, the top $\to \infty$ and the bottom $\to \infty$, so we have “$\frac{\infty}{\infty}$” and we can use L'Hospital's rule.

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{H}{=} \lim_{x\to\infty} \frac{((\ln x)^2)'}{(x)'} = \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x}$$

Now this limit is simpler than the one we started with, but we still can't evaluate it immediately because, as $x \to \infty$ the top and bottom still both go to infinity. We try using L'Hospital's rule again:

$$\lim_{x\to\infty} \frac{2\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{(2\ln x)'}{(x)'} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$$

So putting it all together,

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = 0$$

5.) Evaluate the following limit:

$$\lim_{x\to 0}(\csc x - \cot x)$$

As $x \to 0$, $\csc x = \frac{1}{\sin x} \to \infty$ and $\cot x = \frac{\cos x}{\sin x} \to \infty$, so we have: "$\infty - \infty$." We want to rewrite this to get a quotient.

$$\lim_{x\to 0}(\csc x - \cot x) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\frac{1-\cos x}{\sin x}$$

Now notice that as $x \to 0$ the top $\to 0$ and the bottom $\to 0$, so we have: "$\frac{0}{0}$" and we can use L'Hospital's rule:

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x} \overset{H}{=} \lim_{x\to 0}\frac{(1-\cos x)'}{(\sin x)'} = \lim_{x\to 0}\frac{\sin x}{\cos x} = \frac{0}{1} = 0$$

So we can conclude:

$$\lim_{x\to 0}(\csc x - \cot x) = 0$$

Notice that we could have evaluated this limit using the tricks from Chapter 3:

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x} = \lim_{x\to 0}\frac{\frac{1-\cos x}{x}}{\frac{\sin x}{x}} = \frac{0}{1} = 0$$