5.2 The Definite Integral

Math 1271, TA: Amy DeCelles

1. Overview

Definition: The definite integral of $f(x)$ from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ is the length of the subintervals and x_i^* is any sample point in the i^{th} subinterval.

Sample Points: What is the deal with x_i^*'s? Well, x_i^* just stands for any sample point in the i^{th} subinterval. We know that when we take the limit, it won't matter whether we picked right endpoints, left endpoints, or even any random point in the subinterval. So that's why we don't specify exactly what x_i^* is. We could specify the sample points if we wanted to, for example if we chose to use right endpoints then:

$$\begin{aligned}
x_1^* &= x_1 = a + \Delta x \\
x_2^* &= x_2 = a + 2\Delta x \\
x_3^* &= x_3 = a + 3\Delta x \\
x_4^* &= \dots
\end{aligned}$$

On the other hand, if we chose to use left endpoints:

$$\begin{aligned}
x_1^* &= x_0 = a \\
x_2^* &= x_1 = a + \Delta x \\
x_3^* &= x_2 = a + 2\Delta x \\
x_4^* &= \dots
\end{aligned}$$

When we take the limit, it won't matter which sample points we used.

Terminology: integrand, limits of integration, upper limit, lower limit, integral sign, Riemann sum

Note: The definite integral is a *number*.

Note: The area of a region below the x-axis is counted as *negative* when we're computing the definite integral. For example:

$$\int_0^{2\pi} \sin x = 0$$

because the area below the x-axis is counted as negative, and it cancels out the area above the x-axis.

Some Useful Formulas

$$\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Some Properties of the Definite Integral:

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
2. $\int_a^a f(x)\,dx = 0$
3. $\int_a^b c\,dx = c(b-a)$
4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. $\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$
6. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

Comparison Properties:

1. If $f(x) \geq 0$ on $[a,b]$, then
$$\int_a^b f(x)\,dx \geq 0$$
2. If $f(x) \geq g(x)$ on $[a,b]$, then
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$
3. If $m \leq f(x) \leq M$ on $[a,b]$, then
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

2. Examples

1.) Write the limit as a definite integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1+x_i} \quad \text{on } [1,5]$$

We need to figure out what $f(x)$ is, and what a and b are. Since the interval is $[1,5]$, we can see that $a = 1$ and $b = 5$. The thing being added up should be $f(x_i)\Delta x$, so we can see that:

$$f(x_i) = \frac{e^{x_i}}{1+x_i}$$

So that would make

$$f(x) = \frac{e^x}{1+x}$$

So the definite integral is:

$$\int_1^5 \frac{e^x}{1+x}\,dx$$

2.) Evaluate the definite integral

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

Well, we don't have too many methods for evaluating definite integrals right now. Basically we have two options: we could write it out as a limit of Riemann sums and try to evaluate that limit, or we could try to interpret the integral in terms of an area that we already know how to compute. (And we want to avoid Riemann sums if we can.)

Notice that $f(x) = \sqrt{4 - x^2}$ is a semicircle. It has radius 2 and it is centered at the origin, so the integral from -2 to 2 is the area of the whole semicircle:

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = \text{(area of semicircle of radius 2)}$$

Now the area of a semicirle is half the area of a circle πr^2, so:

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = \tfrac{1}{2} \cdot \pi(2)^2 = 2\pi$$

3.) If $\int_0^5 f(x)\, dx = 10$ and $\int_0^{10} f(x)\, dx = 13$, what is $\int_5^{10} f(x)\, dx$?

We know that:

$$\int_0^5 f(x)\, dx + \int_5^{10} f(x)\, dx = \int_0^{10} f(x)\, dx$$

So that means:

$$10 + \int_5^{10} f(x)\, dx = 13$$

So we can conclude that $\int_5^{10} f(x)\, dx = 3$.

4.) Show that $0 \le \int_0^{\pi/4} \sin x\, dx \le \frac{\sqrt{2}\pi}{8}$.

We want to use the comparison properties. We need to find an upper bound and a lower bound for $\sin x$ on the interval $[0, \frac{\pi}{4}]$. Well, we can see from the unit circle:

$$0 \le \sin x \le \sin(\tfrac{\pi}{4}) = \tfrac{\sqrt{2}}{2} \quad \text{for all } x \text{ in } [0, \tfrac{\pi}{4}]$$

So by the comparison properties,

$$0 \le \int_0^{\pi/4} \sin x\, dx \le \int_0^{\pi/4} \tfrac{\sqrt{2}}{2}\, dx$$

And we can evaluate the last integral:

$$\int_0^{\pi/4} \tfrac{\sqrt{2}}{2}\, dx = \tfrac{\sqrt{2}}{2} \cdot (\tfrac{\pi}{4} - 0) = \tfrac{\sqrt{2}\pi}{8}$$

So we can conclude that $0 \le \int_0^{\pi/4} \sin x\, dx \le \frac{\sqrt{2}\pi}{8}$.