Name: ______________________ **Section:** __________

Names of collaborators: ______________________

Main Points:

1. Estimating accumulated (net) change over an interval, the area under a curve
2. Define the definite integral as the exact net change/exact area under curve
3. Properties of the definite integral

1. Estimating accumulated change

If a quantity Q changes at a constant rate r, with respect to time, then the net change ΔQ in Q over the time interval Δt is exactly: $\Delta Q = r \cdot \Delta t$.

Of course, if Q does not change at a constant rate, this is will not work. Suppose that the instantaneous rate of change of Q, with respect to time, is given by $r(t)$. Then we can *estimate* the change in Q over a short time interval $\Delta t = t_2 - t_1$ by: $\Delta Q \approx r'(t_1) \cdot \Delta t$ or $\Delta Q \approx r(t_2) \cdot \Delta t$. We have done this in the section on linear approximation.

If we wish to estimate the change in Q over a longer time interval, we could break the time interval into short pieces, estimate the change in Q over each short time interval, and then add up these small changes. For an example with distance and velocity, see 5.1, Example 4.

Exercises

1. The rate of change of the world's population, in millions of people per year, is given below.

Year	1950	1960	1970	1980	1990	2000
Rate of change	37	41	78	77	86	79

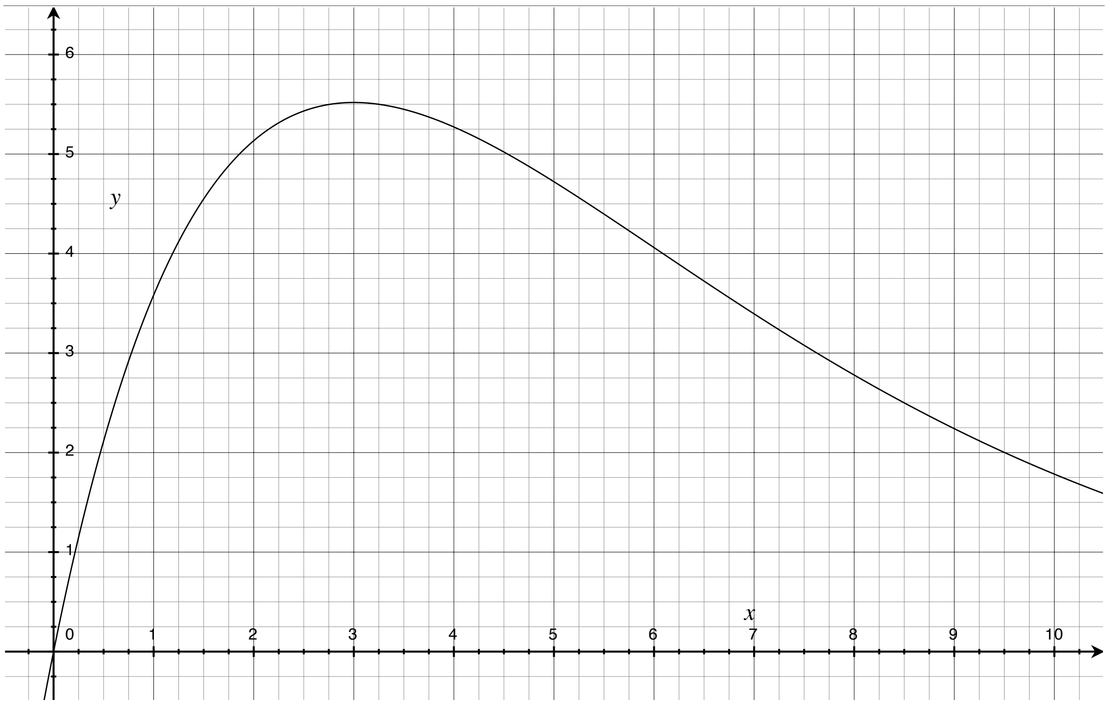
   (a) Use this data to estimate the total change in the world's population between 1950 and 2000.

   (b) The world population was 2555 million people in 1950 and 6085 million people in 2000. Calculate the true value of the total change in the population. How does this compare with your estimate above?

If we have a graph of the rate $r(t)$, then we can see that the multiplication of $r(t_1) \cdot \Delta t$ represents the area of a rectangle of height $r(t_1)$ and width Δt. Thus estimating net change can be understood as estimating the area under a curve using rectangles, whose height is determined by the graph of the function.

Exercise:

2. The velocity (ft/sec) of an object over time (sec) is shown in the graph below.



(a) Estimate the total distance the object traveled between $t = 0$ and $t = 10$, using velocity data every five seconds.

(b) Estimate the total distance the object traveled between $t = 0$ and $t = 10$, using velocity data every two seconds.

3. (See 5.1, Example 1.) Consider the region under the curve $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$.

   (a) Estimate the area using 5 approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is this an over-estimate or under-estimate?

   (b) Estimate the area using 5 approximating rectangles and *left* endpoints. Sketch the graph and the rectangles. Is this an over-estimate or under-estimate?

2. Accumulated change and the definite integral

We find exact accumulated change and exact areas under curves using limits.

To approximate the area under the curve $y = f(x)$ on the interval $[a, b]$ with n rectangles, we divide up the integral into n even subintervals. Each of these little subintervals has length $\Delta x = \frac{b-a}{n}$. This will be the width of the rectangles. We name the left endpoint of the i^{th} interval x_{i-1} and the right endpoint x_i. Then since we start at a and add Δx each time, $x_i = a + i\Delta x$.

We have several options for how high the rectangles should go. For now we just say that x_i^* is a sample point in the i^{th} subinterval (so it's any number between x_{i-1} and x_i). So the area of the i^{th} rectangle will be:

$$A_i = (\text{height of } i^{\text{th}} \text{ rectangle}) \cdot (\text{width of } i^{\text{th}} \text{ rectangle}) = f(x_i^*) \cdot \Delta x$$

So we approximate the area under the curve by adding up the area of the rectangles:

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

This sum is called a Riemann sum.

Taking more and more rectangles (larger and larger n) improves the estimate. The limit as $n \to \infty$ of the Riemann sum is exact accumulated change, or, from a graphical perspective, the exact area under the curve. This is how we define the *definite integral.*

Definition: The definite integral of $f(x)$ from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ is the length of the subintervals and x_i^* is any sample point in the i^{th} subinterval.

Sample Points: Remember that x_i^* stands for any sample point in the i^{th} subinterval. When we take the limit, it won't matter whether we picked right endpoints, left endpoints, or some other points. That's why it is not necessary to specify exactly what x_i^* is, in the definition of the definite integral.

Negative Area? If a rate function $r(t)$ is negative, this means that the quantity function $Q(t)$ is decreasing and the net change over a time interval is negative. Because of this, when we talk about the definite integral as the "area under the curve" we really mean that it is the *signed* area between the curve and the x-axis: the area is positive when the curve is above the x-axis and the area is negative when the curve is below the x-axis.

Exercises

4. Read Note 1 (p 372) on terminology.

   (a) What is the symbol $\int$ called? __________________________

   (b) In the integral $\int_1^5 \sin x\,dx$, what is $\sin x$ called? __________________________

   (c) In the integral $\int_1^5 \sin x\,dx$, what are 1 and 5 called, together? separately?

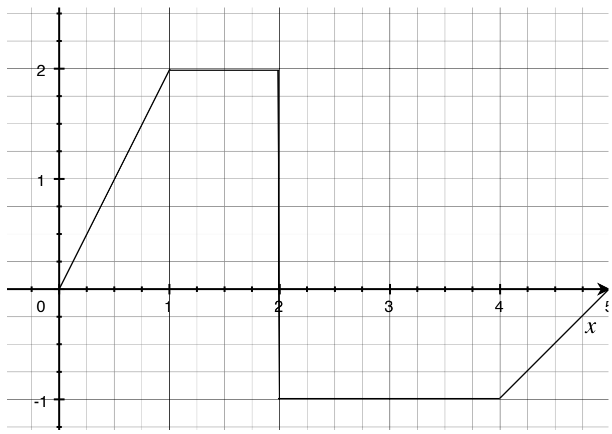
5. Consider the constant function $f(x) = 5$.

   (a) The integral $\int_0^3 f(x)\,dx$ represents the area under the curve from $x = 0$ to $x = 3$. Graph the function on the interval $[0, 3]$. What is the value of the integral?

   (b) Suppose b is a positive number. What is the value of the integral $\int_0^b f(x)\,dx$?

   (c) Suppose $a < b$. What is the value of the integral $\int_a^b f(x)\,dx$?

6. Consider the function $f(x)$ whose graph is below, and evaluate the given integrals, using the fact that the integral is defined as the (signed) area between the curve and the x-axis.



(a) $\int_0^1 f(x)\,dx \quad =$

(b) $\int_0^2 f(x)\,dx \quad =$

(c) $\int_1^4 f(x)\,dx \quad =$

(d) $\int_2^5 f(x)\,dx \quad =$

7. Evaluate the definite integral $\int_{-2}^{2} \sqrt{4 - x^2}\,dx$. (Hint: you need to recognize $y = \sqrt{4 - x^2}$ as the equation of a figure whose area you know from geometry. See 5.2 Example 4.)

3. Properties of the Definite Integral

Read pages 379-381 about the properties of the definite integral and fill in the blanks:

- ____________________ $= \; -\int_b^a f(x)\,dx$
- $\int_a^a f(x)\,dx \;=$ __________
- $\int_a^b c\,dx \;=$ ____________________
- $\int_a^b [f(x) \pm g(x)]\,dx \;=$ ____
- $\int_a^b c \cdot f(x)\,dx \;=$ ____________________
- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx \;=$ ____________________
- If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \;\geq$ __________
- If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \;\geq$ ____________________
- If $m \leq f(x) \leq M$ on $[a, b]$, then ____________________ $\leq \; \int_a^b f(x)\,dx \;\leq$ ____________________

Exercises

8. If $\int_0^5 f(x)\,dx = 10$ and $\int_0^{10} f(x)\,dx = 13$, what is $\int_5^{10} f(x)\,dx$?

9. Suppose that a function $f(x)$ satisfies $2 \;\leq\; f(x) \;\leq\; 3$ on the interval $[1, 4]$. What is the largest that $\int_1^4 f(x)\,dx$ can be? The smallest?