Math 113, Prep: Antiderivatives and the Fundamental Theorem of Calculus (4.9, 5.3)

Name: ____________________ **Section:** ________

Names of collaborators: ____________________

Main Points:

1. The accumulated net change function ("area-so-far" function)
2. Connection to antiderivative functions: the Fundamental Theorem of Calculus
3. Evaluating definite integrals using antiderivatives

1. The accumulated net change function or "area-so-far" function

Suppose we have a rate function $r(t)$ for a quantity Q, and we want to know the accumulated net change in Q for many different time intervals. We fix a given intial time t_0 and let $A(t)$ be the net change from t_0 to t, for many different t-values. Graphically, this is represented by finding the (signed) area between the graph of r and the t-axis from t_0 to t. So $A(t)$ is sometimes called the "area-so-far" function. We know that this area is represented by a definite integral: $\int_{t_0}^{t} r(s)\,ds$.

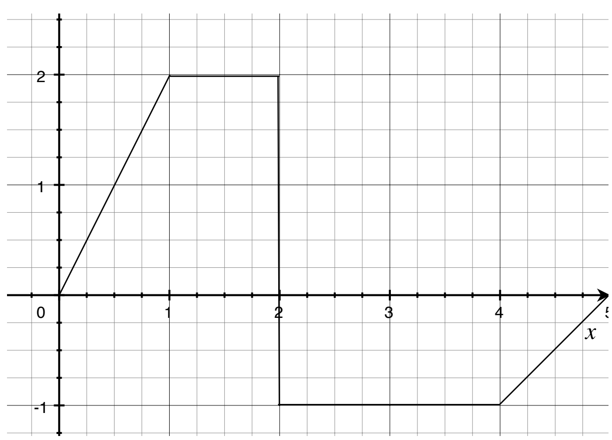
In general, given a function $f(x)$ we may consider the (signed) area between the graph of f and the x-axis from an inital x-value, say a to an arbitrary x. Then the area-so-far function is

$$A(x) \quad = \quad \int_a^x f(t)\,dt$$

Note: we cannot write $f(x)$ in the integral, because we have already chosen x to represent the variable for the area-so-far function, and we cannot use it to mean two different things in the same expression. This is why we write $f(t)$ instead. (See 5.3 Example 1.)

Exercise

1. Given the function $f(x)$ below, and an inital x-value of $x = 0$, we define the area-so-far function to be $A(x) = \int_0^x f(x)\,dx$.



(a) Evaluate $A(x)$ for the x-values given below:

x	0	1	2	3	4	5
$A(x)$						

(b) On what x-interval(s) is $A(x)$ increasing? Why?

(c) At what x-value(s) does $A(x)$ have a local maximum? Why?

2. Accumulated change and antiderivative functions

Suppose that we have a rate function $r(t)$ for a quantity Q, and we want a formula for $Q(t)$. If we know the initial quantity Q_0 and we know the net change ΔQ from the initial time t_0 to time t, then we can find $Q(t)$ simply by adding:

$$Q(t) \quad = \quad Q_0 \ + \ \Delta Q \quad = \quad Q_0 \ + \int_{t_0}^{t} r(s)\, ds \quad = \quad Q_0 + A(t)$$

where $A(t)$ is the area-so-far function with starting point t_0.

Exercises

2. A honeybee population starts with 100 bees and increases at a rate of $r(t)$ bees per week. Write an formula for the size P of the honeybee population t weeks later. (Use an integral!)

3. A bicyclist is pedaling along a straight road for one hour with a velocity $v(t)$ km/hr. She starts out five kilometers from the lake.

   (a) Write an integral to represent the net change in the bicyclist's position (i.e. her displacement) during her ride.

   (b) Write a formula for her position at time t, using an integral.

Since $r(t)$ is the derivative of $Q(t)$, $Q(t)$ is an antiderivative of $r(t)$. Notice that every choice of initial value Q_0 will give an antiderivative for $r(t)$. In particular, choosing $Q_0 = 0$ shows that the area-so-far function itself is an antiderivative for $r(t)$. In other words $A'(t) = r(t)$. This is the idea behind the first part of the Fundamental Theorem of Calculus.

Further, the accumulated net change in Q over any time interval $[t_1, t_2]$ can be written in terms of the area-so-far function:

$$\Delta Q = \int_{t_1}^{t_2} r(t)\,dt = \int_{t_0}^{t_2} r(t)\,dt - \int_{t_0}^{t_1} r(t)\,dt = A(t_2) - A(t_1)$$

This is the idea behind the second part of the Fundamental Theorem of Calculus.

Exercises

4. State the Fundamental Theorem of Calculus, Part 1. (See p 388).

5. Read Examples 2 and 3 in Section 5.3, and find the derivatives of the following area-so-far functions:

   (a) $A(x) = \int_5^x \sin t\, \cos t\, dt$

   (b) $A(t) = \int_1^t s^2 \ln(s)\, ds$

6. State the Fundamental Theorem of Calculus, Part 2. (See p 391.)

7. Read Examples 5, 6, and 7 in Section 5.3. Use FTC 2 to evaluate the definite integrals:

   (a) $\int_0^4 e^x\, dx$

   (b) $\int_1^2 3x^2\, dx$

   (c) $\int_0^{\pi/2} \cos x\, dx$

3. Evaluating definite integrals using antiderivatives

Sometimes we can find a simple formula for an antiderivative, simply by recognizing the given function as a derivative. For example, an antiderivative for $\cos x$ is $\sin x$, an antiderivative for $3x^2$ is x^3, etc. Sometimes, it takes a little work to recognize the given function as a derivative. Read p 344-347 in Section 4.9, up to and including Example 4.

Exercises

8. Find the most general form of the antiderivative of:

   (a) $f(x) = e^x + x^2$

   (b) $g(x) = \dfrac{1}{\sqrt{1-x^2}}$

   (c) $h(x) = \sec^2 x + \sqrt{x}$

9. Evaluate the definite integrals.

   (a) $\int_1^4 (5 - 2t + 3t^2)\, dt$

   (b) $\int_0^{\pi/4} (2 \sec\theta \tan\theta + \cos\theta)\, d\theta$

10. Sometimes it is necessary to rewrite a function in order to recognize it as a derivative, as in Example 4.9.2. For each of the following functions, rewrite the function and then find the most general form of the antiderivative:

    (a) $f(x) = \dfrac{x^2 + x + 1}{x}$

    (b) $h(t) = \dfrac{\sin t}{\cos^2 t}$

11. Evaluate the definite integrals:

    (a) $\int_0^2 (y-1)(2y+1)\,dy$

    (b) $\int_0^4 (4-t)\sqrt{t}\,dt$

12. The velocity (in m/s) of a falling object is given by the formula $v(t) = 5 - 10t$.

    (a) Find the displacement of the object during the first 5 seconds.

    (b) Find a formula for the position function $s(t)$ in terms of the initial position s_0.

13. Find the signed area between the curve $y = x^{-2/3}$ and the x-axis from $x = 1$ to $x = 8$.