Name: ____________________ **Section:** ________

Names of collaborators: ____________________

Main Points:

1. Sums, differences, products, quotients, and compositions of functions
2. Evaluating limits using limit laws
3. Evaluating limits using the Squeeze Theorem

1. Combining Functions

It is relatively straightforward to combine functions by adding them, subtracting them, multiplying them, and dividing them. For example if $f(x) = x^2$ and $g(x) = e^x$,

$$\begin{aligned}
(f+g)(x) &= x^2 + e^x \\
(f-g)(x) &= x^2 - e^x \\
(f \cdot g)(x) &= x^2 e^x \\
\left(\frac{f}{g}\right)(x) &= \frac{x^2}{e^x} = x^2 e^{-x}
\end{aligned}$$

To combine functions by **composition**, however, is a bit more subtle. The composition $f \circ g$ of two functions is obtained by applying the two functions one after the other in this order:

$$x \mapsto g(x) \mapsto f(g(x))$$

Using the example functions above:

$$x \mapsto e^x \mapsto (e^x)^2$$

Since $(e^x)^2 = e^{2x}$, we have

$$(f \circ g)(x) = f(g(x)) = (e^x)^2 = e^{2x}$$

Exercises

1. With $f(x) = x^2$ and $g(x) = e^x$, as above, find a formula for $g \circ f$.

2. With $f(x) = \frac{1}{x-1}$ and $g(x) = \cos(x)$, find formulas for $f \circ g$ and $g \circ f$.

3. The function $F(x) = \sin(x^2)$ can be written as the composition $f \circ g$ of two functions $f(x)$ and $g(x)$. Find formulas for f and g. Also find a formula for $g \circ f$.

4. Express the function $G(x) = \sqrt{\frac{x+1}{x-1}}$ as the composition $f \circ g$ of two functions $f(x)$ and $g(x)$. Then find a formula for $g \circ f$.

2. Evaluating Limits Using Limit Laws and Other Theorems

So far, we have estimated imits by looking at graphs and tables, and we have discussed infinite limits, but we have not actually computed exact values for any finite limits. The good news is that the functions in our catalogue of functions are very well-behaved, and often we can evaluate limits simply by plugging in a number.

Read the eleven limit laws on pages 99-101 of the textbook. These limit laws tell us how to compute limits of combinations of functions, as long as we know the limits of the basic pieces making up the combination. In particular, a nice consequence is the **Direct Substitution Property** for polynomial and rational functions.

If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

i.e. the limit can be evaluated by simply plugging in the number.

The catch is that the number a must be in the domain of f, and a rational function often has some numbers that are not in its domain. See Example 3 on page 102.

Exercises: Compute the following limits:

5. $\displaystyle\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4}$

6. $\displaystyle\lim_{x\to -1} \frac{x^2+5x+4}{x^2+3x-4}$

7. $\displaystyle\lim_{x\to -1} \frac{x^2-4x}{x^2-3x-4}$

8. $\displaystyle\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$ (Hint: Rationalize the numerator, as in Example 6, page 103.)

For functions involving absolute value it is helpful to notice that taking the absolute value of a negative number is the same as multiplying by (-1), e.g. $|-4| = (-1)(-4)$, so

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Sometimes it is necessary to compute left and right limits separately, for example for piecewise functions or functions involving an absolute value. If the left and right limits agree, say both are equal to a number L, then the limit itself is L. See Examples 7 and 8, page 104.

The **Squeeze Theorem** can also be useful for evaluating limits:

If $f(x) \leq g(x) \leq h(x)$ near a (except possibly at a), and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then,

$$\lim_{x\to a} g(x) = L$$

i.e. if the function $g(x)$ is sandwiched between two functions that have the same limit at a certain point, then the limit of $g(x)$ at that point must also be the same.

See Example 11, page 105.

Exercises:

9. Evaluate the limit: $\displaystyle\lim_{x \to -2} \frac{2 - |x|}{2 + x}$

10. Suppose that, for all x, $2x \le g(x) \le x^4 - x^2 + 2$. What can we say about $\displaystyle\lim_{x \to 1} g(x)$?