Name: ____________________ **Section:** ________

Names of collaborators: ____________________

Main Points:

1. Careful definition of continuity using limits
2. Theorem about continuity of some familiar functions
3. Intermediate Value Theorem (IVT)

1. Definition of Continuity

Recall our intuitive notion of continuity: a function $f(x)$ is continuous at a point $x = a$ if $f(x)$ is close to $f(a)$ when x is close to a. For example, we are implicitly using the continuity of $f(x) = \sqrt{x}$ at $x = 4$, when we say that $\sqrt{4.1}$ should be about $\sqrt{4}$ (i.e. 2), since 4.1 is close to 4.

Now that we discussed the notion of a limit, we can use limits to make this intuitive notion more precise:

A function f is continuous at a number a if: $\lim_{x\to a} f(x) = f(a)$

Notice that this definition implicitly requires three things:

(1) $f(a)$ is defined

(2) $\lim_{x\to a} f(x)$ exists

(3) $\lim_{x\to a} f(x)$ equals $f(a)$

Note that (2) implicitly requires three things as well: that the limit from the left exists, the limit from the right exists, and both of these limits agree.

So, to show that a function is continuous at a given point, we must show that (1), (2), and (3) are all true.

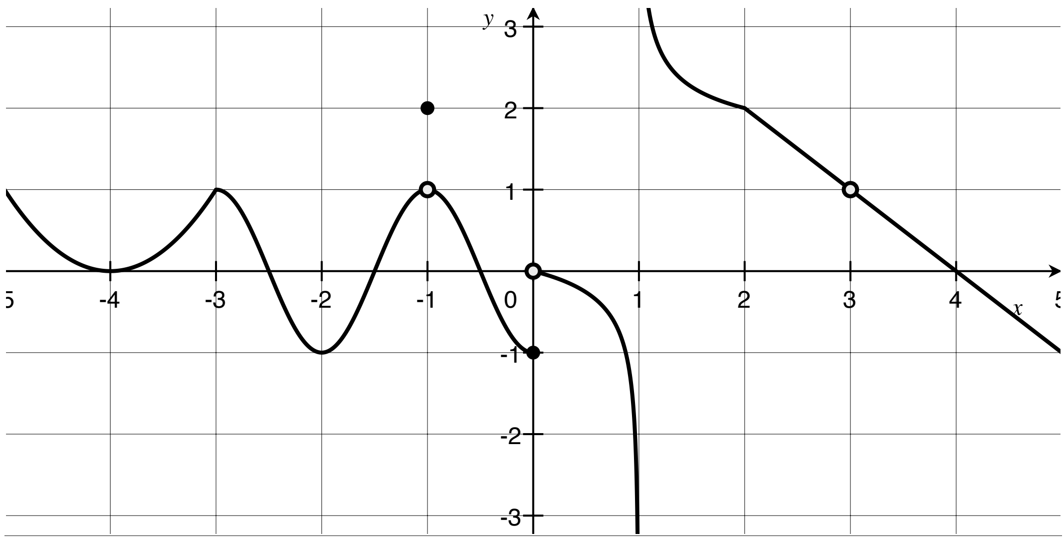
To show that a function is *discontinuous* at a given point, we must demonstrate that one or more of the conditions (1), (2) or (3) is violated. (See Examples 1 and 2 on page 119.)

See the first full paragraph on page 120 for a description of three kinds of discontinuities: removable discontinuities, infinite discontinuities, and jump discontinuities.

We say that a function is continuous on an interval (a, b) if it is continuous for all numbers in the interval. In particular, a function is continuous on $(-\infty, \infty)$ if it is continuous for all real numbers. In this case, we say that it is continuous (period).

Exercises

1. Consider the function $f(x)$ in the graph below, and fill in the table to describe the behavior of the function at the specified points. Write "DNE" if the specified value does not exist. In the column where you are asked "why", state which condition(s), (1), (2), or (3) from above, is violated. In the last column, classify the discontinuity as a removable discontinuity, an infinite discontinuity, or a jump discontinuity.



x_o	$\lim_{x\to x_o^-} f(x)$	$\lim_{x\to x_o^+} f(x)$	$\lim_{x\to x_o} f(x)$	$f(x_o)$	Continuous? Y/N	If no, why?	What kind?
-3							
-2							
-1							
0							
1							
2							
3							
4							

2. Use the definition of continuity to show that function $f(x) = x^2 + 2$ is continuous at $x = -2$. (You need to check (1), (2), and (3).)

3. Use the definition of continuity to show that function below is continuous at $x = \pi/4$. (Again, check (1), (2), and (3). This time you will need to use left and right limits to find the limit as $x \to \pi/4$.)

$$f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$$

4. Use the definition of continuity to explain why the function below is discontinuous at $x = -2$.

$$f(x) = \begin{cases} \dfrac{(x+2)}{(x+2)(x-1)} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$$

(Hint: You need to use the conditions (1), (2), and (3) again. To show that the function is *not* continuous at $x = -2$, you need to show that one of the conditions is violated. This requires up to three steps: first find $f(-2)$, then compute the limit as $x \to -2$, then compare the two and draw your conclusion.)

5. We can change the function from the previous exercise to make it continuous, by changing the 1 to another number. What number c would make the function continuous at $x = -2$?

$$f(x) = \begin{cases} \dfrac{(x+2)}{(x+2)(x-1)} & \text{if } x \neq -2 \\ c & \text{if } x = -2 \end{cases}$$

This is called "removing the discontinuity."

6. Consider the function $f(x) = \dfrac{1-x^2}{2-x-x^2}$.

   (a) Use the definition of continuity to explain why $f(x)$ is discontinuous at $x = -2$. (This may require three steps, but think about whether you really need all three steps in this case.)

   (b) Use the definition of continuity to explain why $f(x)$ is discontinuous at $x = 1$. (Again, think about whether you really need all three steps in this case.)

   (c) What kind of discontinuity (jump, removable, or infinite) is at $x = -2$? How can you tell? (Hint: factor the numerator and denominator.)

   (d) What kind of discontinuity (jump, removable, or infinite) is at $x = 1$? How can you tell?

2. Continuity of Some Familiar Functions

The familiar functions that we have discussed are important both because they model real-life phenomena that we care about and because they are relatively well-behaved. In particular, they don't have jump discontinuities. Some of them (for example, many rational functions) do have removable discontinuities and/or infinite discontinuities, as in the exercise above. However, these discontinuities only occur at numbers that are not in the domain of the original function. In other words:

> **Theorem:** All polynomials, rational functions, root functions, trig functions, inverse trig functions, exponential functions, and logarithmic functions are continuous where defined.

Look over pages 121-123 for a justification of this theorem.

Exercise:

7. Show that the function below is continuous on $(-\infty, \infty)$.

$$f(x) \quad = \quad \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$$

(Hint: You need to show that this function is continuous for every real number a. To do this, break it down into three cases: (i) $a < \pi/4$, (ii) $a > \pi/4$, and (iii) $a = \pi/4$. For the first two cases, use the theorem above. For the third case, just reference Exercise 3!)

2. The Intermediate Value Theorem

Common sense tells us that if some quantity, say temperature, changes in a continuous way with an initial value of, say 52°, and a final value of, say 81°, then for any value between those the inital and final values, say a temperature of 68°, there must have been a moment when the quantity was exactly equal to that intermediate value, 68°.

This idea is formalized in the **Intermediate Value Theorem (IVT)**:

> Suppose f is continuous on a closed interval $[a, b]$. Suppose the y-values $f(a)$ and $f(b)$ are not equal. Pick any number N between $f(a)$ and $f(b)$. Then there is a number c in (a, b) such that $f(c) = N$.

Notice that the IVT does *not* tell you *anything* about the number c, except that it is between a and b! So you should not expect to find c, unless you are *specifically* asked to find it.

The IVT is often used to show that a given equation has a root in between two specified values. See Example 10, page 126.

Exercise:

8. Use the IVT to show that there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.