Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. Definition of derivative as limit of difference quotients
2. Interpretation of derivative as slope of graph
3. Interpretation of derivative as instantaneous rate of change

1. Limit of Difference Quotients

Recall from the preparatory assignment on limits (2.1, 2.2), that to estimate the slope of a tangent line, we use the slopes of secant lines. In particular, to estimate the slope of the tangent line to the graph of a function $f(x)$ at a point $x = a$ in the domain of f, the slope of the secant line through $P = (a, f(a))$ and $Q = (a+h, f(a+h))$ is:

$$D \;=\; \frac{\Delta y}{\Delta x} \;=\; \frac{f(a+h)\;-\;f(a)}{h}$$

This is called the *difference quotient*. The value of D depends on h.

The limit of $D(h)$ as h approaches zero (if the limit exists) is the slope of the tangent line. In the preparatory assignment on limits, you explored this idea numerically, by creating tables of D-values for smaller and smaller values of h.

Definition For a function $f(x)$ and a number a in its domain, the derivative of f at a, denoted $f'(a)$, is:

$$f'(a) \;=\; \lim_{h\to 0} D(h) \;=\; \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

provided that the limit exists.

Exercise

1. Look back at your preparatory exercises for 2.1 and 2.2 to answer these questions.

   (a) If $f(x) = \sqrt{x}$ and $a = 4$, what is a good estimate for $f'(a)$?

   (b) If $f(x) = |x^2 - 4|$ and $a = 2$, what can you say about $f'(a)$?

2. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = 5$ if $f(5) = -3$ and $f'(5) = 4$.

3. Suppose that $y = 4x - 5$ is an equation of the tangent line to the curve $y = f(x)$ at the point $x = 2$. Find $f(2)$ and $f'(2)$.

Having discussed how to evaluate limits symbolically (as in 2.3), we can determine the exact value for $f'(a)$, at least in some relatively simple examples. (See Example 4 on page 147.)

Exercise

4. Let $f(x) = \sqrt{x}$ and $a = 4$. Find the exact value of $f'(a)$ by evaluating the following limit symbolically as in Section 2.3:
$$\lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$$
(You will need to rationalize the numerator, as in Section 2.3, Example 6, page 103.)

2. Interpretation as Rates of Change

So far we have been thinking of the difference quotient from a geometric point of view, as the slope of a secant line. In applications, we interpret slopes and difference quotients as rates of change.

For example, on the very first day of class we modeled the dependency of the chirp rate of crickets on the temperature by a linear function. The slope of the function is

$$\frac{\text{change in chirp rate (chirps/min)}}{\text{change in temperature (°F)}} \quad = \quad 3.75 \;\; \text{(chirps/min)/°F}$$

This is the rate at which the chirp rate changes with respect to the temperature. This means that when the temperature increases by one degree, the chirp rate will increase by 3.75 chirps/min, according to our model. Since our model is linear, the rate of change is constant. (The slope of a straight line is constant.)

In general, the rate of change will vary. In such cases it is necessary to distinguish average rates of change from instantaneous rates of change. One important example is velocity: the rate of change of position with respect to time.

Suppose $s(t)$ represents the position of an object at a given time. Then the average velocity of the object is a difference quotient, which can be written in several different ways:

$$v_{\text{ave}} \quad = \quad \frac{\Delta s}{\Delta t} \quad = \quad \frac{s(t_1) - s(t_0)}{t_1 - t_0} \quad = \quad \frac{s(t_0 + h) - s(t_0)}{h}$$

The instantaneous velocity at time $t = t_0$ is:

$$v \quad = \quad \lim_{t_1 \to t_0} \frac{s(t_1) - s(t_0)}{t_1 - t_0} \quad = \quad \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h} \quad = \quad s'(t_0)$$

Thus the derivative of the position function at $t = t_0$ is the instantaneous velocity at t_0.

In general, if $Q(x)$ represents a quantity, depending on another quantity x, then the average rate of change of Q with respect to x is given by a difference quotient:

$$r_{\text{ave}} \quad = \quad \frac{\Delta Q}{\Delta x} \quad = \quad \frac{Q(x_1) - Q(x_0)}{x_1 - x_0} \quad = \quad \frac{Q(x_0 + h) - Q(x_0)}{h}$$

The instantaneous rate of change when $x = x_0$ of $Q(x)$ with respect to x is:

$$r \quad = \quad \lim_{x_1 \to x_0} \frac{Q(x_1) - Q(x_0)}{x_1 - x_0} \quad = \quad \lim_{h \to 0} \frac{Q(x_0 + h) - Q(x_0)}{h} \quad = \quad Q'(x_0)$$

Thus the derivative of of $Q(x)$ at $x = x_0$ is the instantaneous rate of change of Q with respect to x.

In summary, we can interpret the difference quotient and the derivative in these ways:

difference quotient	derivative
slope of secant	slope of tangent
average velocity	instantaneous velocity
average rate of change	instantaneous rate of change

For examples of interpretating difference quotients and derivatives, see Examples 6 and 7, p 148-149.

Exercises

5. Let $T(t)$ be the temperature (in °F) in Phoenix t hours after midnight on September 10, 2008. The table shows values of this function recorded every two hours.

t	0	2	4	6	8	10	12	14
T	82	75	74	75	84	90	93	94

(a) How much did the temperature change from 6 am to 8 am? What is the average rate of change of the temperature over this time interval? Make sure to include units in your answer.

(b) How much did the temperature change from 8 am to 10 am? What is the average rate of change of the temperature over this time interval? Make sure to include units in your answer.

(c) Estimate $T'(8)$ by averaging your results from (a) and (b).

(d) What is the meaning of $T'(8)$? Make sure to include units in your answer.

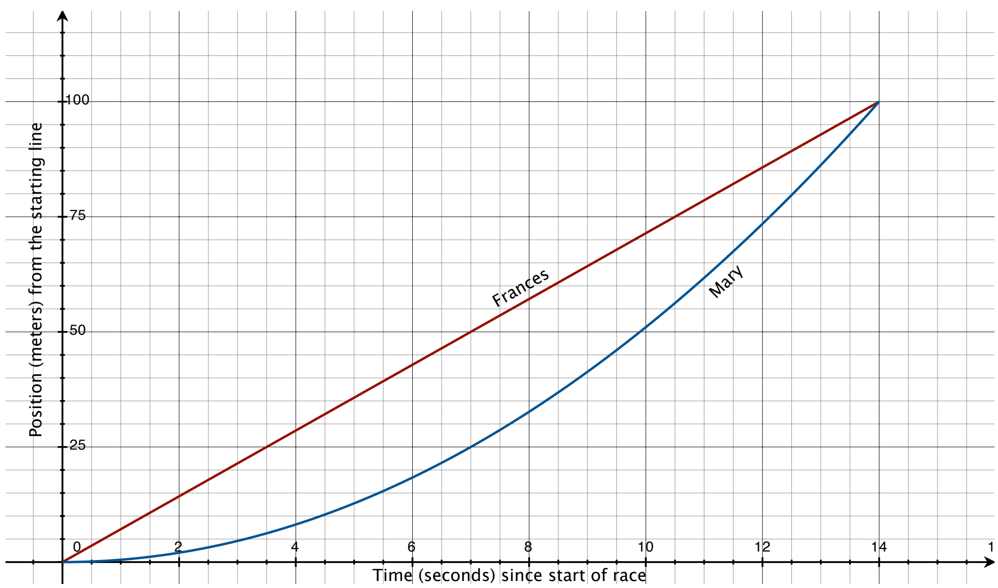
(e) Use (d) to estimate the temperature at 9:00 am.

6. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.

   (a) What is the meaning of the statement $f'(5) = 100$? What are the units of $f'(5)$?

   (b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? Why?

7. Shown are graphs of the position functions of two runners, who run a 100 meter race.

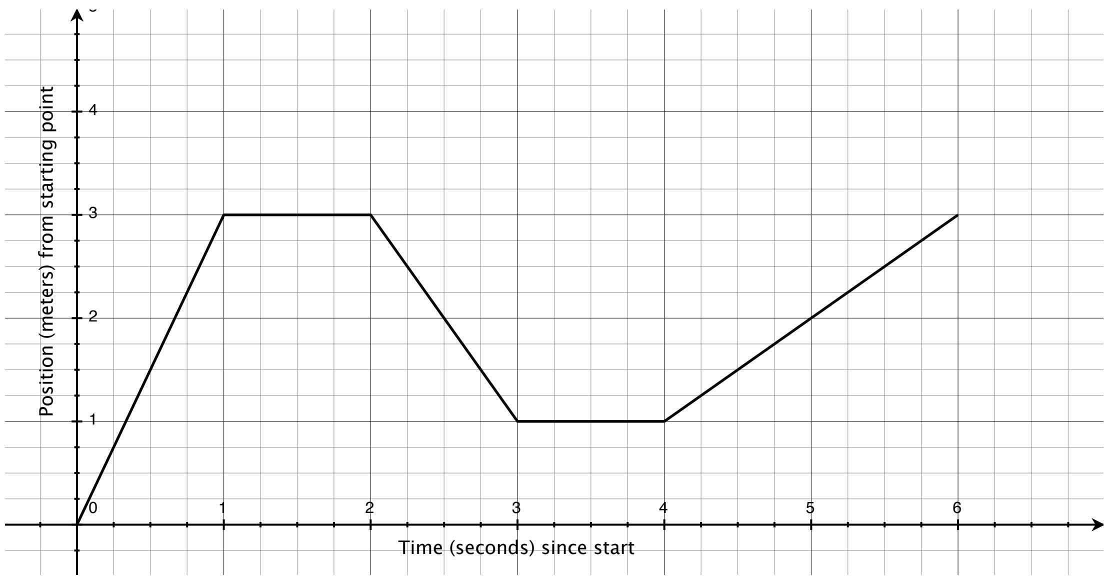


   (a) Describe and compare how the runners run the race. (In particular, who won the race?)

   (b) At what time is the distance between the runners the greatest? __________

   (c) At what time do they have the same velocity? __________

8. A duck starts by waddling north on 35E; the graph of its position function is shown.



(a) When is the duck waddling north? ____________________

(b) Waddling south? ____________________

(c) Standing still? ____________________

(d) Draw a graph of the velocity function on the axes below.