Math 113, Prep: Implicit Differentiation and Logarithmic Differentiation (3.5B, 3.6B)

Name: ______________________ **Section:** ________

Names of collaborators: ______________________

Main Points:

1. Implicit differentiation
2. Logarithmic differentiation

1. Implicit Differentiation

So far we have looked at finding $\frac{dy}{dx}$ when y is defined explicitly by a function of x, i.e. $y = f(x)$. Now we will look at finding $\frac{dy}{dx}$ when the relationship between x and y might not be so simple. For example, we might have an equation with x's and y's on both sides, and it might not be possible to get the y on a side by itself. This means that y is defined *implicitly*. The method of finding $\frac{dy}{dx}$ in such a case is called *implicit differentiation*. See Examples 1-3 in 3.5.

Implicit Differentiation:

1. Take $\frac{d}{dx}$ of both sides.
2. Take derivatives remembering that y is a function of x. Use the product rule, quotient rule, chain rule where appropriate. For example:

   Product Rule: $\frac{d}{dx}(\sin x)(y) = (\cos x)(y) + (\sin x)(\frac{dy}{dx})$

   Quotient Rule: $\frac{d}{dx}(\frac{x^2}{y}) = \frac{2xy - x^2(\frac{dy}{dx})}{y^2}$

   Chain Rule: $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$

3. Get $\frac{dy}{dx}$ on a side by itself.

Exercises

1. In this problem, we will practice using the product rule, quotient rule, and chain rule, outside of an implicit differentiation problem. Find the following derivatives, assuming that y depends on x.

   (a) $\frac{d}{dx}\ y\ \cos(x)$

   (b) $\frac{d}{dx}\ \sqrt{x}\, y$

(c) $\frac{d}{dx}\ \frac{\sin(x)}{y}$

(d) $\frac{d}{dx}\ \tan(y)$

(e) $\frac{d}{dx}\ \ln(y)$

(f) $\frac{d}{dx}\ x^2\, y^3$

(g) $\frac{d}{dx}\ y\, e^{x^2}$

2. Use implicit differentiation to find $\frac{dy}{dx}$ when $4x^2 + 9y^2 = 36$.

3. Find y' when $y^5 + x^2 y^3 = 1 + ye^{x^2}$

4. Find the tangent line to the curve $\sqrt{x} + \sqrt{y} = 4$ at the point $(4, 4)$.

2. Logarithmic Differentiation

Logarithmic differentiation is a technique we apply to particularly nasty functions when we want to differentiate them.

Recall the laws of logarithms:

- $\log(ab) = \log a + \log b$
- $\log(a/b) = \log a - \log b$
- $\log(a^r) = r \log a$

In particular, notice that the log takes a *product* and gives us a *sum*, and when it comes to taking derivatives, we like sums better than products! Similarly, a log takes a *quotient* and gives us a *difference*. Again, when it comes to taking derivatives, we'd much prefer a difference to a quotient. Finally, the log takes something of the form a^b and gives us a product. Again, this is an improvement when it comes to differentiation.

See 3.6, Examples 7 and 8.

Logarithmic Differentiation: Suppose you have $y = f(x)$ and $f(x)$ is a nasty combination of products, quotients, etc. The trick is to:

- Apply the natural log to both sides: $\ln y = \ln(f(x))$, and use the laws of logs to simplify the right hand side as much as possible.
- Take the derivative (with respect to x) of both sides. You have to use the chain rule on the left side:

$$\frac{y'}{y} = (\text{RHS})'$$

- Solve for y' by multiplying both sides by the original function:

$$y' = f(x) \cdot (\text{RHS})'$$

Exercises

1. Consider the function $y = \sqrt{x}\, e^{x^2}(x^2+1)^{10}$.

   (a) Apply the natural log to both sides and use the log laws to simplify the right hand side, namely $\ln\left(\sqrt{x}\, e^{x^2}(x^2+1)^{10}\right)$, as much as possible.

   (b) Apply $\frac{d}{dx}$ to both sides, use implicit differentiation, and solve for y'.

2. Find the derivative y' of $y = x^{1/x}$.

3. In this exercise, we look at four cases for exponents and bases.

   (a) Consider the function $f(x) = x^{9/2}$. What kind of function is this? What is the name of the differentiation rule that is used to find the derivative? What is $f'(x)$?

   (b) Consider the function $g(x) = (5/4)^x$. What kind of function is this? What is $f'(x)$?

   (c) Consider the function $h(x) = \pi^e$. What kind of function is this? What is $h'(x)$?

   (d) Consider the function $j(x) = x^x$. Is this function in our catalog of familiar functions? What technique can be used to differentiate $j(x)$? What is $j'(x)$?

4. Find the derivatives.

   (a) $\frac{d}{dx}\left(1+2x\right)^{9/2}$

(b) $\frac{d}{dx}\,(5/4)^{1/x}$

(c) $\frac{d}{dx}\,(\sin x)^{x}$