Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________________

Main Points:

1. Using L'Hospital's rule to evaluate limits of quotients
2. Variations on L'Hospital's Rule

1. Limits of Quotients

This section gives us a way to evaluate limits that look like "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$". The trick is to use L'Hospital's rule (pronounced low-pea-tahl, not la-hospital), which says that you can take the derivative of the top and the derivative of the bottom and *then* take the limit of *that*. More precisely:

> **L'Hospital's Rule**: Suppose $F(x)$ is a quotient $F(x) = \frac{f(x)}{g(x)}$ (where f and g are differentiable and $g'(x) \neq 0$ near a, except possibly at a.) Suppose that one of the following is true:
>
> Case 1: ("$\frac{0}{0}$") $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$
> Case 2: ("$\frac{\infty}{\infty}$") $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$
>
> Then we can replace f/g with f'/g' in the limit:
>
> $$\lim_{x\to a} F(x) = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$
>
> as long as the limit on the RHS exists (or is $\pm\infty$.)

Note: This also works for right/left hand limits and limits at infinity.

Note: Do *not* confuse this with the quotient rule! We are *not* taking the derivative of $F(x)$ when we apply L'Hospital's rule. We do take the derivative of the top and the derivative of the bottom in order to evaluate the limit.

Exercises:

1. Evaluate the limit $\lim_{x\to -2} \dfrac{x+2}{x^2+3x+2}$ in two ways:

   (a) using the methods of Chapter 2. (Factor and cancel!)

   (b) using L'Hospital's Rule.

2. Evaluate the limit $\lim_{x\to\infty} \dfrac{3x^2+2}{x^2+3x+2}$ in two ways:

   (a) using the methods of Chapter 2. (Divide top and bottom by x^2!)

   (b) using L'Hospital's Rule. (You need to use it twice.)

3. Evaluate the limit $\lim_{x\to 0} \dfrac{x+\tan x}{\sin x}$ in two ways:

   (a) using the methods of Chapter 3 (i.e. use the fact that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.)

   (b) using L'Hospital's Rule

4. Evaluate the limit: $\lim_{x\to\infty} \dfrac{(\ln x)^2}{x}$. (See Examples 1 and 2.)

2. Variations: Indeterminate Products, Differences, and Powers

The following variations are useful.

- Intederminate Products "$0 \cdot \infty$": Write $f(x)g(x)$ as a quotient:

$$f(x)g(x) \quad = \quad \frac{f(x)}{\frac{1}{g(x)}} \quad \text{or} \quad \frac{g(x)}{\frac{1}{f(x)}}$$

  Then use L'Hospital's rule. (See Example 6.)

- Inteterminate Differences "$\infty - \infty$": Rewrite as quotient using a common denominator, rationalization, or factoring out a common factor. Then use L'Hospital's rule. (See Example 7.)
- Indeterminate Powers "0^0", "∞^0", or "1^∞": Use the natural log.

Exercises:

5. Evaluate the limits by rewriting as a quotient and using L'Hospital's Rule:

   (a) $\lim_{x \to \infty} x\, e^{-x}$

   (b) $\lim_{x \to 0} (\csc x - \cot x)$

6. Evaluate the limits, using L'Hospital's Rule if appropriate:

   (a) $\lim_{x \to 2} \dfrac{x-2}{x^2+2}$

   (b) $\lim_{x \to \infty} \sqrt{x}\, e^{-x/2}$

(c) $\lim_{x \to 0} x\, e^x$

(d) $\lim_{x \to 0^+} \dfrac{x}{\ln x}$

7. Show that $\lim_{x \to \infty} \dfrac{e^x}{x^n} = \infty$, for any positive integer n. What does this tell you about exponential growth as compared polynomial growth?