Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. Approximating slope of tangent line by slopes of secant lines, slope function (difference quotient), approximating instantaneous rates of change using average rates of change
2. Estimating limits from tables or graphs, right and left limits
3. Determining infinite limits, vertical asymptotes

1. The Slope Function

Consider a function $f(x)$ and a number a in its domain. We wish to estimate the slope of the tangent line to the graph of $f(x)$ at $x = a$. We can estimate the slope of the tangent line using secant lines. For a small number h, the slope of the secant line through $P = (a, f(a))$ and $Q = (a + h, f(a + h))$ is:

$$D = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

This is called the *difference quotient*. The value of D depends on h. When we consider D as a function of h, we call it the *slope function* $D(h)$. As h gets closer and closer to zero, the x-value of the point Q gets closer to the x-value of the point P.

Exercises

1. Let $f(x) = \sqrt{x}$ and $a = 4$. We wish to approximate the slope of the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

   (a) Use *Mathematica* to compute the D-values, and record your answers, keeping six digits, in the following tables.

h	D
1	
0.1	
0.01	
0.001	

h	D
-1	
-0.1	
-0.01	
-0.001	

   (b) What trends do you notice? What happens to D as h gets close to zero? Does it matter whether h is positive or negative?

(c) According to your computations in (a), what is the slope of the secant line through the points $(4,2)$ and $(4.1, \sqrt{4.1})$? (No new computations are needed.)

(d) Estimate the slope of the tangent line at $x = 4$.

2. Now let $f(x) = |x^2 - 4|$ and $a = 2$.

(a) Use *Mathematica* to compute the D-values, and record your answers, keeping six digits, in the following tables.

h	D
1	
0.1	
0.01	
0.001	

h	D
-1	
-0.1	
-0.01	
-0.001	

(b) What trends do you notice? What happens to D as h gets close to zero? Does it matter whether h is positive or negative?

2. The Limit of a Function

The formal mathematical notion of a *limit* is used to describe trends like the ones observed above. Intuitively, if the values of $f(x)$ approach a specific number L, as x approaches a from either side, then that number L is called the limit. (Read the first few paragraphs on page 87.)

Definition Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of $f(x)$, as x approaches a, equals L"

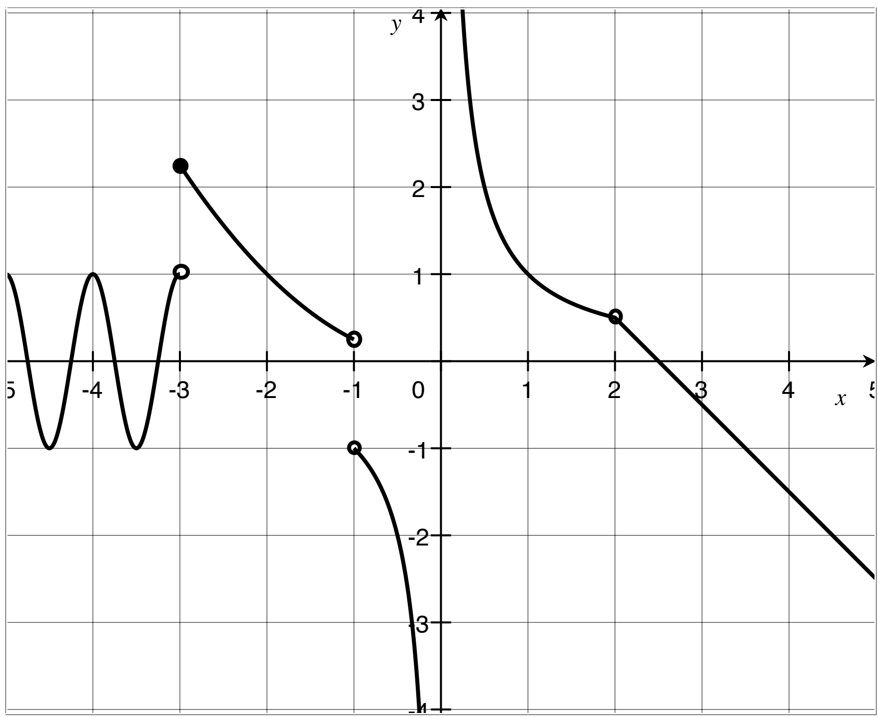
if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x sufficiently close to a (on either side of a) but not equal to a.

Similarly we can talk about the limit from the left and the limit from the right, if we only mean to discuss a approaching x from the left, or right, respectively. (See Example 7, page 92.)

When a function increases without bound (informally: the values "go to infinity") we use the infinity symbol (∞) to denote the limit, even though the limit technically does not exist, because the values of the function to not approach a specific number. (See Example 8, page 93.)

Exercises

3. Consider the function $f(x)$ in the graph below, and fill in the table to describe the behavior of the function at the specified points. Write "DNE" if the specified value does not exist.



x_o	$\lim_{x \to x_o^-} f(x)$	$\lim_{x \to x_o^+} f(x)$	$\lim_{x \to x_o} f(x)$	$f(x_o)$
-3				
-1				
0				
2				
4				

4. Let $f(x) = \dfrac{\sin x}{x}$ and $a = 0$.

   (a) Use *Mathematica* to fill in the tables below, with numbers accurate to six digits.

x	$f(x)$
1	
0.1	
0.01	

x	$f(x)$
-1	
-0.1	
-0.01	

   (b) Use your work in (a) to estimate the following limits, if they exist. If a limit does not exist, write "DNE".

$\lim_{x\to 0^-} f(x) =$ __________ $\lim_{x\to 0^+} f(x) =$ __________ $\lim_{x\to 0} f(x) =$ __________

   (c) What is $f(0)$? What does this tell you about what the graph looks like near $x = 0$?

5. Let $f(x) = \dfrac{x}{x-1}$ and $a = 1$.

   (a) Use *Mathematica* to fill in the tables below, with numbers accurate to six digits.

x	$f(x)$
2	
1.1	
1.01	
1.001	

x	$f(x)$
0	
0.9	
0.99	
0.999	

   (b) Use your work in (a) to estimate the following limits, if they exist. If a limit does not exist, write "DNE".

$\lim_{x\to 1^-} f(x) =$ __________ $\lim_{x\to 1^+} f(x) =$ __________ $\lim_{x\to 1} f(x) =$ __________

   (c) What is $f(1)$? What does this tell you about what the graph looks like near $x = 1$?

6. Let $f(x) = \dfrac{x^2 - 1}{x - 1}$ and $a = 1$.

   (a) Use *Mathematica* to fill in the tables below, with numbers accurate to six digits.

x	$f(x)$
2	
1.1	
1.01	
1.001	

x	$f(x)$
0	
0.9	
0.99	
0.999	

   (b) Use your work in (a) to estimate the following limits, if they exist. If a limit does not exist, write "DNE".

$\lim_{x\to 1^-} f(x) =$ ________ $\lim_{x\to 1^+} f(x) =$ ________ $\lim_{x\to 1} f(x) =$ ________

   (c) What is $f(1)$? What does this tell you about what the graph looks like near $x = 1$?

7. Look back at your work with the slope functions and write your conclusions using the limit notation:

   (a) For $f(x) = \sqrt{x}$, $a = 4$:

$\lim_{h\to 0^-} D(h) =$ ________ $\lim_{h\to 0^+} D(h) =$ ________ $\lim_{h\to 0} D(h) =$ ________

   (b) For $f(x) = |x^2 - 4|$, $a = 2$:

$\lim_{h\to 0^-} D(h) =$ ________ $\lim_{h\to 0^+} D(h) =$ ________ $\lim_{h\to 0} D(h) =$ ________

3. Determining Infinite Limits and Vertical Asymptotes from Formulas

We explored the concept of infinite limits above, using graphs and tables. We can also determine some infinite limits symbolically. (Read Examples 9 and 10 on page 95.) In stating vertical asymptotes, remember that the equation of a vertical line is of the form $x = a$, for some constant a.

Exercises

8. Evaluate the following infinite limits symbolically:

   (a) $\lim_{x \to 5^-} \frac{6}{x-5}$

   (b) $\lim_{x \to 3} \frac{2-x^3}{(3-x)^2}$

   (c) $\lim_{x \to \pi^-} \csc x$

9. State the equations of the vertical asymptotes of the functions in the previous problem:

   (a) ______________ (b) ______________ (c) ______________