Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. Linear Approximation
2. Differentials

1. Linear Approximation

We have learned how to find the tangent line to a curve at a specific point. If the function is $f(x)$, and the point is $(a, f(a))$, the equation for the tangent line is:

$$y - f(a) = f'(a) \cdot (x - a)$$

$$y = f'(a)(x - a) + f(a)$$

We now give the tangent line a new name: the *linearization* of f at a and we call it $L(x)$:

$$L(x) = f'(a)(x - a) + f(a)$$

We can use the tangent line to approximate $f(x)$ when x is near a:

$$f(x) \approx L(x) = f'(a)(x - a) + f(a) \quad \text{when } x \text{ is close to } a$$

This approximination is called the *linear approximation* or *tangent line approximation* of f at a. See Example 1, page 251.

Exercises

1. The number of active Facebook users hit 175 million at the end of February 2009 and 200 million at the end of April 2009. With t in months since the start of 2009, let $f(t)$ be the number of active users in millions.

   (a) What is $f(4)$? Interpret your answer in terms of Facebook users. (Write a complete sentence, and make sure to include units.)

   (b) Estimate $f'(4)$. Interpret your answer in terms of Facebook users. (Write a complete sentence, and make sure to include units.)

(c) Estimate $f(5)$, using (a) and (b). Interpret your answer in terms of Facebook users. (Write a complete sentence, and make sure to include units.)

(d) What is the linearization of f at $a = 4$?

2. Let $f(x) = \sqrt{x+7}$.

(a) Find the linearization $L(x)$ of f at $a = 2$.

(b) Use the linearization you found to estimate the value of f of at $a = 3$.

(c) Using (b), estimate $\sqrt{10}$. What approximation does your calculator give for $\sqrt{10}$? How close is your estimate to the calculator's?

3. Find the linearization of $f(x) = (x+2)^5$ at $a = 0$, and use it to approximate $(2.001)^5$.

2. Differentials

Another way to think about this is in terms of *differentials*. Let's say we have a function $f(x)$ and we can easily compute the value at a. We are wondering what happens to $f(x)$ if we move away from a just a little bit. We call the "little bit" Δx or dx. So the question is, What is $f(a + \Delta x)$? We would expect that it would not be very different from $f(a)$, since Δx is small, but we call the difference Δy:

$$\Delta y = f(a + \Delta x) - f(a)$$

We will approximate Δy in the following way. When we are looking at the point a, the function is changing at a rate of $f'(a)$. So if we multiply the rate (a.k.a. the slope of the tangent) by the interval Δx (which is the same as dx) that will give us an approximation for Δy:

$$\Delta y \approx f'(a) \cdot dx$$

We define dy to be $f'(a) \cdot dx$ and we call dx and dy differentials. See Example 3, page 253.

Exercises

4. For some painkillers, the size of the dose given, D, depends on the weight of the patient, W. Thus $D = f(W)$, where D is in miligrams an W is in pounds.

   (a) Interpret the statements $f(140) = 120$ and $f'(140) = 3$ in terms of this painkiller. (Use complete sentences, and include units.)

(b) Use the information in the statements part (a) to estimate $f(145)$.

(c) Now use differentials to describe your estimation in (b). In particular, make sure to state what dW and dD are.

5. Suppose we wish to approximate $1/4.002$ using differentials. We will look at the function $f(x) = 1/x$ near $a = 4$.

   (a) What is $f(a)$? __________ Δx? __________ dx? __________

   (b) What is dy?

   (c) Estimate Δy.

   (d) Estimate $1/4.002$ using your work above. What estimate does your calculator give for $1/4.002$? How close is your estimate to the calculator's?

6. Use differentials to estimate $e^{-0.015}$. (In your set-up, make sure to state $f(x)$, a, Δx, and dx.)

In experiments, we often measure certain quantities and calculate other quantities from our measurements. We can use differentials to estimate the error in the calculated quantity, given the error in the measured quantities. See Example 4, page 254. Read the note after Example 4 for an explanation of *relative error* and *percentage error*.

7. Suppose we measure the radius of a disk to be 24 cm, with a maximum error of .2 cm. Then we use our measurement for the radius to compute the area. Estimate the maximum error in the calculated area. What is the relative error? What is the percentage error?