Name: ______________________________

Names of collaborators: ______________________________

0.1 Main Points to Review:

1. Functions, models, graphs, tables, domain and range
2. Algebraic functions, power functions, exponential functions, trig functions
3. One-to-one functions, inverse functions, logarithms, inverse trig functions

0.2 Functions

A function is a relationship between two quantities, in which one quantity (e.g. position within this room) uniquely determines the other (e.g. air temperature). Given a position in the room, the air temperature is uniquely determined. Notice that the inverse relationship *cannot* be modeled with a function: we cannot consider position as a function of air temperature, because there may be many positions in the room that have the same air temperature. Said differently, the air temperature does not uniquely determine a position in the room.

There are (at least) four ways to represent a function: verbal (words), numerical (table), graphical, and symbolic (formula).

Exercise: Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

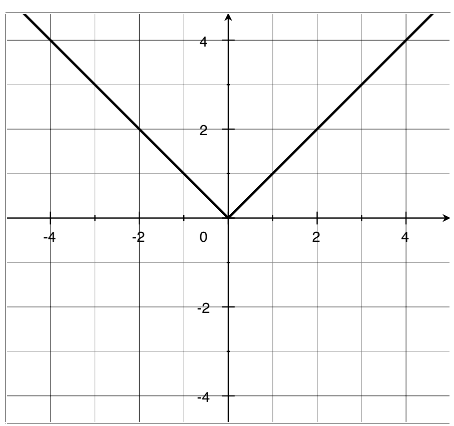

Figure 1: Is this function of x? Y/N

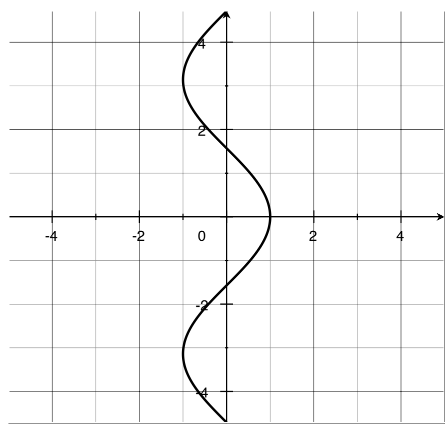

Figure 2: Is this function of x? Y/N

0.3 Algebraic Functions and Power Functions

Linear functions, and polynomial functions more generally, are very familiar and well-behaved. In particular, the domain of a polynomial is all real numbers. Rational functions are ratios of polynomials. These are relatively well-behaved but may have domain restrictions and/or asymptotes. In particular, if the denominator is zero for a specific x-value, that x-value is excluded from the domain.

Power functions are of the form $f(x) = x^a$, where a is a nonzero constant. There is quite a variety of functions in this family, including:

1. familiar monomials: x^2, x^3, x^4 etc.
2. root functions: $x^{1/2} = \sqrt{x}$, $\quad x^{1/3} = \sqrt[3]{x}$, $\quad x^{1/4} = \sqrt[4]{x}$, etc.
3. the reciprocal function: $x^{-1} = \frac{1}{x}$

Note: Some of these functions have domain restrictions and/or asymptotes!

We use the general term "algebraic function" to refer to any function that is a combination of polynomial, rational, and root functions.

Exercise: Find the domain of the functions. Write your answer in interval notation.

(a.) $f(x) = 2/(3x - 1)$

(b.) $g(x) = \sqrt{16 - x^4}$.

0.4 Exponential Functions

Exponential functions come up frequently in modeling real world phenomena. Examples include population growth and radioactive decay. Exponential functions are characterized in the following way: given a certain unit of time (say one day) the amount of something (say the number of bacteria in a petrie dish) will increase (or decrease) by the same factor or ratio. So in the case of bacteria, we might say that the population doubles each day. This is exponential growth. If we start with 100 bacteria, the amount of bacteria would be modeled by the following equation:

$$P(t) \quad = \quad 100 \cdot 2^t$$

where $P(t)$ is the number of bacteria after t days.

Exponential decay is the reverse: the amount of a certain radioactive substance decreases by the same ratio every year, say by one-half. Then if we start with 100 grams, the next year there will be 50 grams, then 25 grams, etc. Here the amount of radioactive substance would be modeled by

$$A(t) \quad = \quad 100 \cdot (1/2)^t \quad = \quad 100 \cdot 2^{-t}$$

where $A(t)$ is the amount of radioactive substance left after t years.

There are a few things to notice about exponential functions $f(x) = a^x$ from their graphs:

1. They have only positive outputs.
2. The line $y = 0$ is a horizontal asymptote.
3. If $a > 1$, the function is increasing constantly; if $0 < a < 1$ it is decreasing constantly.

Laws of exponents: for $a > 0$ and any real numbers x and y,

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = \frac{a^x}{a^y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

Exercise A bacterial culture starts with 500 bacteria and doubles in size every half hour.

(a) How many bacteria are there after 3 hours? Make a table to show how the amount of bacteria increases.

(b) Find a formula to express how many bacteria there are after t hours.

(c) How many bacteria are there after 40 minutes?

(d) Use a graphing tool (e.g. *Mathematica* or a graphing calculator) to graph the population function and estimate the time for the population to reach 100,000. Sketch the graph below to support your answer.

0.5 Trigonometric Functions

See Appendix D for a thorough review of trigonometry. It is especially important to be familiar with sine, cosine, and tangent. In particular, know the values of these three functions at the five standard angles in the first quadrant: $0, \pi/6, \pi/4, \pi/3, \pi/2$, and how to use these special values to find the values of the other trig functions in all four quadrants.

Exercise: Fill in the following table, using the five standard angles in the first quadrant.

Angle, θ		$\sin\theta$	$\cos\theta$	$\tan\theta$
deg	rad			

Evaluate the function at the given value.

$$\cos(-120^\circ) = \underline{\qquad\qquad\qquad}$$

$$\csc\left(\frac{4\pi}{3}\right) = \underline{\qquad\qquad\qquad}$$

0.6 One-to-one Functions and Inverse Functions

As discussed above in the air temperature example, not every function has a well-defined inverse. However, some functions do. For example, if a population is constantly increasing over time, then, given a population size, say 10 billion people, we could say exactly when that size was attained, e.g. 1971. When a given output ($f(x)$-value) uniquely determines an input (x-value), then the function is *one-to-one*.

Every one-to-one function has an inverse function, i.e. a function that switches inputs and outputs. (You can tell that a function is one-to-one if it passes the horizontal line test.) Inverses *undo* eachother, i.e. if we input x to a function f, and get y as an output, then we input y into the inverse function f^{-1} of f, then we get x back as the output; we're back where we started. In other words: if $f(x) = y$, then $f^{-1}(y) = x$.

If you have a one-to-one function, you can find its inverse by solving for the independent variable.

The graph of f^{-1} is obtained by reflecting the graph of f through the line $y = x$. (You're basically just switching the role of x and y.)

Exercise: Determine whether each function is one-to-one. If it is one-to-one, sketch a graph of its inverse on the same set of axes.

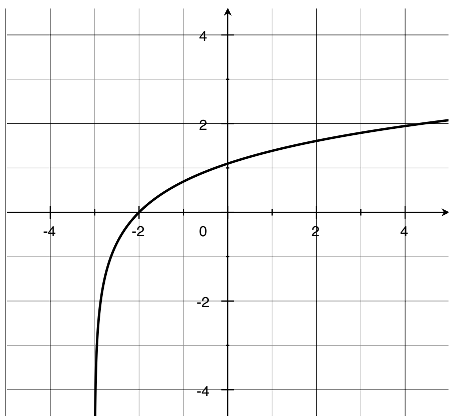

Figure 3: Is this function one-to-one? Y/N

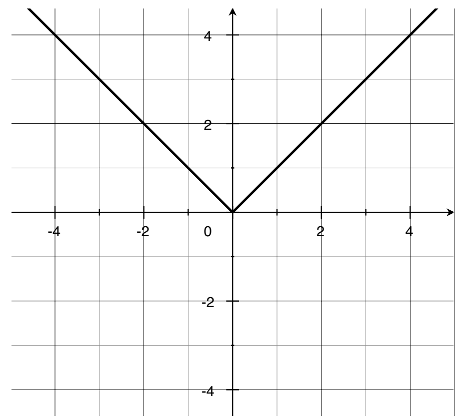

Figure 4: Is this function one-to-one? Y/N

0.7 Logarithms and Inverse Trig Functions

Logarithms are the inverse functions for exponential functions. For a number $b \neq 0$, we define the logarithmic function with base b by:

$$\log_b(x) = y \qquad \iff \qquad b^y = x.$$

The natural logarithm, denoted ln, is the logarithm base e. Notice that since the exponential functions always have positive outputs, the logarithms only make sense for positive inputs. Thus the domain of the function $f(x) = \log_b(x)$ is $(0, \infty)$.

Laws of logarithms: For $b \neq 0$, x, y positive real numbers, and r any real number,

1. $\log_b(xy) = \log_b(x) + \log_b(y)$
2. $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$
3. $\log_b(x^r) = r \log_b(x)$

Another fact that sometimes comes in handy is the change of base formula:

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

Inverse Trig functions:

$$\begin{array}{rclcrclcl}
\sin^{-1}(x) & = & y & \iff & \sin(y) & = & x & \text{and} & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
\cos^{-1}(x) & = & y & \iff & \cos(y) & = & x & \text{and} & 0 \leq x \leq \pi \\
\tan^{-1}(x) & = & y & \iff & \tan(y) & = & x & \text{and} & -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{array}$$

Exercises:

1. Without using a calculator, find the exact value of
   (a) $\ln(1/e)$
   (b) $\log_2 6 - \log_2 15 + \log_2 20$

2. Express the given quantity as a single logarithm: $\ln 5 + 5 \ln 3$.

3. Find the domain of $\ln(x + 6)$.

4. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is given by $P(t) = 100 \cdot 2^{t/3}$.

   (a) Find the inverse of this function and explain its meaning. (Hint: Solve for t in terms of P.)

   (b) When will the population reach 50,000?

5. Without using a calculator, find the exact value of

   (a) $\arctan(1)$

(b) $\cos^{-1}(1/\sqrt{2})$

6. Simplify the expression $\cos(\sin^{-1} x)$. (Hint: Draw a triangle and use the Pythagorean Theorem.)