Math 114, Prep 9.1 (Intro to Differential Equations)

Name: ______________________ **Section:** ________

Names of collaborators: ______________________

Main Points:

1. qualitative analysis of differential equations
2. families of solutions of differential equations

1. Qualitative Analysis of Differential Equations

When a quantity A is *directly proportional* to a quantity B, that means that there is a positive constant, say k, such that $A = kB$.

A simple model for population growth operates under the assumption that a population will grow at a rate directly proportional to the size of the population. In other words $\frac{dP}{dt}$ is proportional to P. How would we write this in an equation?

$$\frac{dP}{dt} = k\,P \qquad \text{(for some } k > 0\text{)}$$

This is an example of a differential equation. Any function $P(t)$ that satisfies this equation is called a *solution* to the differential equation. What can we determine about P without a formula for P? It turns out that we can determine quite a lot!

Exercises.

1. Consider the differential equation modeling population growth given above.

   (a) If $P = 0$, what is $\frac{dP}{dt}$?

   Explain what this means in terms of the population growth by finishing the following sentence: "If the population is zero, its growth rate is . . . , which means . . . ."

   (b) If $P > 0$, what can you say about $\frac{dP}{dt}$?

   Explain what this means in terms of the population growth by finishing the following sentence: "If the population is . . . , its growth rate is . . . , which means . . . ."

   (c) If $P < 0$, what can you say about $\frac{dP}{dt}$?

   Explain what this means in terms of the population growth by finishing the following sentence: "If the population is . . . , its growth rate is . . . , which means . . . ." (Does this make sense?)

2. Read p 581 for a more refined model of population growth, one which takes into account the fact that there are limited resources, and a population cannot grow indefinitely.

   (a) Write down the differential equation that models population growth under ther assumption that there are limited resources.

   (b) When is $\frac{dP}{dt} = 0$? Interpret this in terms of population growth.

   (c) When is $\frac{dP}{dt} > 0$? Interpret this in terms of population growth.

   (d) When is $\frac{dP}{dt} < 0$? Interpret this in terms of population growth.

   (e) Suppose $k = 3$ and $M = 100$. Sketch the graphs of several functions $P(t)$ that satisfy the differential equation. (See Figure 3.) What are the equillibrium solutions?

(f) (Challenge!) Graph $\frac{dP}{dt}$ as a function of P (instead of as a function of t). (This should be a parabola.) At what P-value does $\frac{dP}{dt}$ have a maximum? What does this mean in terms of population growth?

2. Families of solutions

A differential equation relates an unknown function and one or more of its derivatives. A solution to a differential equation is a function that satisfies the differential equation. Read about "General Differential Equations" on pages 582-584.

Exercises

3. Consider the differential equation $x^2 y' + xy = 1$.

(a) Show that every member of the family $y = (\ln(x) + C)/x$ is a solution of this differential equation. (See Example 1.)

(b) Illustrate by graphing several members of the family of solutions on a common screen. (Use *Mathematica* or a graphing calculator.) Sketch your results below.

(c) Find a particular solution that satisfies the initial condition $y(1) = 2$. (See Example 2.)

(d) Find a particular solution that satisfies the initial condition $y(2) = 1$.

4. Consider the differential equation $y' = xy^3$.

   (a) What can you say about the graph of a solution when x is close to 0?

   What if x is large?

   (b) Verify that all members of the family $y = (c - x^2)^{-1/2}$ are solutions of the differential equation.

   (c) Graph several members of the family of solutions on a common screen. Do the graphs confirm what you predicted in part (a)?

   (d) Find a solution of the initial-value problem:

$$y' = xy^3 \qquad y(0) = 2$$