Math 114, Practice for Exam 2

Name: ______________________________

Instructions: The exam will have eight problems. Make sure that your reasoning and your final answers are clear. Include labels and units when appropriate. No notes, books, or calculators are permitted during the exam. The following formulas will be provided.

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$
$$\sin x \, \cos x = \tfrac{1}{2} \sin 2x$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| \qquad \int \tan x\, dx = \ln|\sec x|$$
$$\int \csc x\, dx = \ln|\csc x - \cot x| \qquad \int \cot x\, dx = \ln|\sin x|$$

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1. **Basic Facts and Concepts**

   Fill in the blanks question.

   (a) Average value of a function, its graphical interpretation as the height of a certain rectangle, the MVT for integrals.

   (b) Vocabulary: differential equation, general solution, particular solution.

   (c) Write differential equation from verbal description.

   (d) Parametric curves: how to eliminate a parameter.

   (e) How to convert from polar coordinates to Cartesian coordinates and vice versa.

   (f) Convergence of Sequences of the form $a_n = r^n$ (p 696).

   (*See also the Chapter Review Concept Checks: CR6 #6, CR9 #2, 5, CR10 #2, 4, 5ab, CR11 2*)

2. **Average value of a function and the MVT for Integrals.**

   Consider the function $f(x) = 2x - x^2$ on the interval $[0, 2]$.

   (a) Find the average value of f on the given interval.

   (b) Find c such that $f_{\text{ave}} = f(c)$.

   (c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

   (See also 6.5 #1-15, 19, 21.)

3. **Sample Application of the Integral: Work**

   When a *constant* force F is applied to an object, displacing it by $\Delta x = x_2 - x_1$, the *work* done on the object is defined to be $W = F\,\Delta x$.

   When the force applied to an object *changes* as the object is displaced, an integral is used to compute the work done on the object as it is displaced: $W = \int_{x_1}^{x_2} F(x)\,dx$.

   (a) A 360-lb gorilla climbs a tree to a height of 20 ft. Find the work done if the gorilla reaches that height in (i) 10 seconds and (ii) 5 seconds. (Note: the force required for the gorilla to lift itself is equal to the gorilla's weight.)

   (b) When a particle is located a distance x meters from the origin a force of $\cos(\pi x/3)$ acts on it. How much work is done in moving the particle from $x = 0$ to $x = 1$?

   (*See also 6.4 #1-6.*)

4. **Differential Equations**

(*See also 9.1 #8-15, 9.2 # 2, 9.3 #1-18, 39, 43, CR9 #15*)

Let R be the number of undergraduate students at St. Thomas who have heard a certain rumor t days after the rumor was announced. For the sake of this problem, we will assume that there are 6000 undergraduates at UST. The rate $\frac{dR}{dt}$ at which the rumor spreads with respect to time is *jointly proportional* to the number of students, R, who have heard the rumor and the number of students, $6000 - R$, who have not heard the rumor. This means that the rate is proportional to the product $R(6000 - R)$, with the constant of proportionality being $k = 0.001$. Then we have

$$\frac{dR}{dt} = 0.001\,R(6000 - R)$$

Suppose that the number of people who initially heard the rumor was 30.

For parts (a)-(e) of this problem, you will do a *qualitative analysis* of the differential equation. Answer the questions without finding a formula for the solutions of the differential equation. In your answers for (a), (b), and (c), include all mathematical possibilities, not merely those answers that have a meaningful interpretation in terms of a rumor spreading through the undergraduates at UST.

(a) What are the constant (equilibrium) solution(s) for this equation? ____________________

(b) For what values of R is R increasing? ____________________

(c) For what values of R is R decreasing? ____________________

(d) For what value(s) of R is R increasing most rapidly? What does this mean in terms of the rate that the rumor is spreading?

(e) Make a rough sketch of the solution to this differential equation, given that the initial number of people who heard the rumor is 30. Make sure to label both axes and provide scale for the R axis.

(f) Recall that the spread of the rumor can be modeled by

$$\frac{dR}{dt} = 0.001\, R(6000 - R)$$

Use the technique of separation of variables to find a formula for $R(t)$, given that the initial number of people who heard the rumor is 30. (For this part of the problem, you do not need to find all possible mathematical solutions, just one meaningful solution.)

5. **Parametric Calculus**

   (a) Consider the parametric curve:

   $$x(t) \;=\; 1 + \ln(t) \qquad y(t) \;=\; t^2 + 2$$

   i. Find the tangent line to the curve at the point $(1, 3)$, without eliminating the parameter.
   ii. Now eliminate the parameter, and verify that your answer to (a) is correct by finding the slope of the tangent line again.

   (b) Consider the parametric curve

   $$x(t) \;=\; t^2 + 1 \qquad y(t) \;=\; t^2 + t$$

   For which values of t is the curve concave up?

   (*See also 10.2 #1-20*)

6. **Polar Calculus**

   (a) Find the slope of the tangent line to the curve $r = 4\sin\theta$ when $\theta = \pi/6$.

   (b) Sketch the curve $r = 4\sin\theta$ and find the area it encloses.

   (c) Find the area of the region that lies inside $r = 4\sin\theta$ and outside $r = 2$.

   (*See also 10.3 #15-20, 55-64, 10.4 # 9-12, 23-25, 29-31*)

7. **Sequences**

   (a) List the first five terms of the sequences:

   i. $a_n = \dfrac{3^n}{1+2^n}$

   ii. $a_n = \dfrac{1}{(n+1)!}$

   iii. $a_1 = 2, \quad a_2 = 1, \quad a_{n+1} = a_n - a_{n-1}$

   (b) Determine whether the sequence converges or diverges. If it converges, find the limit.

   i. $a_n = \sqrt{\dfrac{n+1}{9n+1}}$

   ii. $\left\{ \dfrac{e^n - e^{-n}}{e^{2n}-1} \right\}$

   (*See also 11.1 # 1-18, 23-42*)