Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

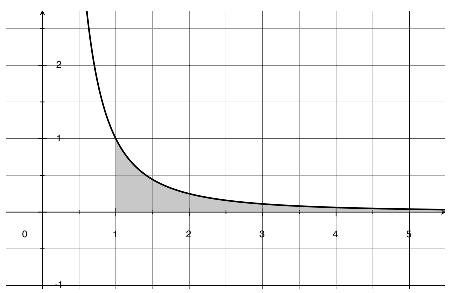
Main Points:

1. Two types of improper integrals
2. An important family of examples: $f(x) = \frac{1}{x^p}$

1. Integrating over infinite intervals

Exercises.

1. Suppose we wish to determine the area of the infinite region bounded by the curves $y = \frac{1}{x^2}$, $x = 1$, and the x-axis:



(Of course, we ought to be wondering whether it is reasonable for an infinite region to have a finite area ...)

(a) Evaluate the following integrals:

i. $\int_1^{10} \frac{1}{x^2}\,dx$

ii. $\int_1^{100} \frac{1}{x^2}\,dx$

iii. $\int_1^{1000} \frac{1}{x^2}\,dx$

(b) Based on your answers in (a), do you think the area of the infinite region described above has a finite area? If so, make a guess for what the area is.

(c) Now consider an arbitrary number T greater than one, and let $A_T = \int_1^T \frac{1}{x^2}\,dx$. (We considered $T = 10, 100, 1000$ above.)

i. What is the value of A_T, for arbitrary T?

$$A_T \;=\; \int_1^T \frac{1}{x^2}\,dx \;=$$

ii. What is the limit of A_T as T tends to infinity?

$$\lim_{T\to\infty} A_T \;=$$

(d) The area of the infinite region is $A = \lim\limits_{T\to\infty} A_T$ from (c)(ii). How does this compare to your guess in (b)?

In general, integrating over infinite intervals is defined by taking a limit after integrating over larger and larger finite intervals, as you did in Exercise 1:

$$\int_a^\infty f(x)\,dx \;=\; \lim_{T\to\infty} \int_a^\infty f(x)\,dx$$
$$\int_{-\infty}^b f(x)\,dx \;=\; \lim_{T\to-\infty} \int_T^b f(x)\,dx$$

Exercises

2. (a) What does it mean for an improper integral to be convergent? Divergent? (See box, p 520.)

(b) Give the definition (from the box on p 520) of the doubly-infinite integral:

$$\int_{-\infty}^{\infty} f(x)\,dx \quad =$$

3. (a) Read Example 1. Is $\int_1^\infty \frac{1}{x}\,dx$ convergent or divergent?

(b) Use a limit to determine whether $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is convergent or divergent. If it converges, what is its value?

(c) Read Example 4, summarize the result, and explain how (a) and (b) fit the pattern.

4. Determine if the integral is convergent or divergent. If convergent, what is its value?

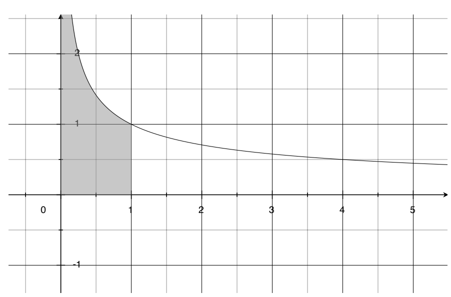
(a) $\displaystyle\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$

(b) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

2. Integrating over discontinuities

Exercises

5. Suppose we wish to determine the area of the infinite region bounded by the curves $y = \frac{1}{\sqrt{x}}$, $x = 1$, and the y-axis:



(a) Evaluate the following integrals:

i. $\int_{0.1}^{1} \frac{1}{\sqrt{x}}\,dx$

ii. $\int_{0.01}^{1} \frac{1}{\sqrt{x}}\,dx$

iii. $\int_{0.001}^{1} \frac{1}{\sqrt{x}}\, dx$

(b) Based on your answers in (a), do you think the area of the infinite region described above has a finite area? If so, make a guess for what the area is.

(c) Now consider an arbitrary number t between zero and one, and let $A_t = \int_t^1 \frac{1}{\sqrt{x}}\, dx$. (We considered $t = 0.1, 0.01, 0.001$ above.)

i. What is the value of A_t, for arbitrary t?

$$A_t = \int_t^1 \frac{1}{\sqrt{x}}\, dx =$$

ii. What is the limit of A_t as t approaches zero from above?

$$\lim_{t\to 0^+} A_t =$$

(d) The area of the infinite region is $A = \lim_{t\to 0+} A_t$ from (c)(ii). How does this compare to your guess in (b)?

In general, an integral over an interval containing a discontinuity is defined by taking a limit, as you did in Exercise 5:

$$\begin{aligned} \int_a^b f(x)\, dx &= \lim_{t\to a^+} \int_t^b f(x)\, dx \qquad \text{(discontinuity at } a) \\ \int_a^b f(x)\, dx &= \lim_{t\to b^-} \int_a^t f(x)\, dx \qquad \text{(discontinuity at } b) \end{aligned}$$

Exercises

6. (a) Suppose $p > 1$. Evaluate $\int_0^1 \frac{1}{x^p}\,dx$, or show that the integral diverges.

   (b) Now suppose $0 < p < 1$. Evaluate $\int_0^1 \frac{1}{x^p}\,dx$ or show that the integral diverges.

   (c) Finally, evaluate the same integral, with $p = 1$, i.e. $\int_0^1 \frac{1}{x}\,dx$, or show that it diverges.

7. If a discontinuity occurs in the middle of the inverval over which we are integrating, special care is needed. Suppose a function $f(x)$ has a discontinuity at a point $x = c$ in the middle of the interval $[a, b]$. Give the definition (from the box on p 523) of the improper integral:

$$\int_a^b f(x)\,dx \quad =$$

8. Evaluate the integral $\int_{-1}^{8} x^{-1/3}\,dx$. (You need to break up the integral into two integrals, and use limits to evaluate each integral, as in Example 7.)