Math 114, Prep 11.4 (Limit Comparison Test)

Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. comparing with familiar series, whose convergence/divergence is known
2. making a careful argument and invoking the Limit Comparison Test

The Limit Comparison Test can be used in some cases where the Comparison Test cannot be used. For example, we may *suspect* that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - n - 1}$$

converges, since it is similar to a p-series with $p = 2$. However, the Comparison Test cannot be used to prove this, since the terms of this series are not actually *smaller* than the terms of this p-series. We will use the Limit Comparison Test instead.

Exercises.

1. Read pages 724-725. State the Limit Comparison Test (middle of page 724.)

2. In this exercise, we will use the Integral Test to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - n - 1}$ converges, by comparison with the p-series, with $p = 2$.

   (a) (**Set-up**) Define the series $\sum a_n$ to be the given series and $\sum b_n$ to be the other (more familiar) series, whose convergence/divergence is known.

   (b) (**Check Hypotheses**) Check to make sure the series $\sum a_n$ and $\sum b_n$ satisfy the hypothesis in the Limit Comparison Test, namely that they are series with positive terms.

(c) (**Set up the ratio and take the limit.**) Set up the ratio a_n/b_n and simplify. Then take the limit as $n \to \infty$.

$$\frac{a_n}{b_n} =$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} =$$

(d) Is this limit a finite, positive number? __________

If so, proceed to the next step, if not you need to find a different sequence b_n.

(e) (**Discuss known series**) Explain how you know that the more familiar series $\sum b_n$ converges or diverges.

(f) (**Apply Test and Draw Conclusion**) Use the Limit Comparison Test to conclude that the series $\sum a_n$ converges (or diverges), and state your conclusion in a sentence. Make sure to include the phrase "by the Limit Comparison Test" somewhere in your sentence.

3. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + n + 1}$.

(a) Do you think this series converges or diverges? (It diverges.) What known (divergent) series can you compare it to? Will the ratio of terms a_n/b_n converge to a positive, finite number?

(b) Give a careful argument, using the Limit Comparison Test, to prove that the series diverges. (Your argument should follow the outline given in the previous problem.)

4. Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n-1}}$.

   (a) Do you think this series converges or diverges? What known series can you compare it to? Will the ratio of terms a_n/b_n converge to a positive, finite number? (Hint: Remember that $n\sqrt{n} = n^{3/2}$.)

   (b) Give a careful argument, using the Limit Comparison Test, to prove that the series converges (or diverges.)

5. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$.

   (a) Do you think this series converges or diverges? What known series can you compare it to? Will the ratio of terms a_n/b_n converge to a positive, finite number? (Hint: Notice that $e^{1/n} \to e^0 = 1$ as $n \to \infty$.)

   (b) Give a careful argument, using the Limit Comparison Test, to prove that the series converges (or diverges.)

6. Look back at the four series above. Could you have used the Comparison Test instead of the Limit Comparison Test for any of them? Which one(s)?