Name: ______________________________ **Section:** __________

Names of collaborators: ______________________________

Main Points:

1. horizontal and vertical tangents, concavity
2. arclength and area

1. Derivatives: tangents and concavity

For a curve in the xy-plane, the slope of a tangent line (assuming there is a well-defined tangent line!) to the curve at the point (a,b) is $\frac{dy}{dx}|_{(a,b)}$. Last semester, we learned how to find $\frac{dy}{dx}$ when the curve is in Cartesian form, using the derivative of a function or using implicit differentiation. In this section we discuss how to find $\frac{dy}{dx}$ for a curve in parametric form. We use the Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{y'(t)}{x'(t)} \qquad (\text{if } x'(t) \neq 0)$$

Note that this will give us a formula for $\frac{dy}{dx}$ in terms of t.

To determine concavity, use the second derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Read Examples 1 and 2.

Exercises.

1. Find dy/dx if $x = 1/t$ and $y = \sqrt{t}\, e^{-t}$.

2. Find an equation of the tangent line to the curve $x = 1 + 4t - t^2, y = 2 - t^3$ at the point corresponding to $t = 1$.

3. Find an equation of the tangent to the curve $x = 1 + \sqrt{t}, y = e^{t^2}$ at the point $(2, e)$ by two methods:

   (a) without eliminating the parameter.

   (b) by first eliminating the parameter.

4. Consider the curve $x = e^t$, $y = te^{-t}$.

   (a) Find dy/dx.

   (b) Find d^2y/dx^2. For which values of t is the curve concave upward?

2. Integrals: area and arclength

Remembering that $dx = x'(t)dt$, we can write the area under a parametric curve as:

$$A = \int_{x=a}^{x=b} y\,dx = \int_{t=\alpha}^{t=\beta} y(t)\,x'(t)\,dt$$

The arclength of the parametric curve is

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

as long as the curve is traversed exactly once as t increases from α to β. Read Examples 3, 4, and 5.

Exercises.

5. Consider a and b to be fixed positive constants. Sketch the curve $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$, and find the area that it encloses. (Hint: Use symmetry. Divide the area into four equal pieces and find the area of one piece, then multiply by four to get the total area.)

6. Find the exact length of the curve $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \le t \le 2$.