Name: ______________________________ **Section:** ________

Names of collaborators: ______________________________

Main Points:

1. representing functions with power series
2. interval of convergence

1. Power Series

As we have discussed a couple times this semester, not all functions that turn out to be interesting or useful for applications can be described in terms of familiar functions.

For example, the function e^{-x^2} is used in probability, but its antiderivative is not elementary: it cannot be expressed in terms of familiar functions. It is called the "error function," sometimes denoted erf.

Another example would be a "Bessel function," which is a solution to the differential equation $x^2y'' + xy' + x^2y = 0$, which is used to model electromagnetic waves, heat conduction, and vibrating membranes. Bessel functions also come up as solutions to the Schrödinger equation for a free particle.

One way to represent such functions is as "power series," which can be thought of as polynomials with infinitely many terms:

$$c_0 \;+\; c_1x \;+\; c_2x^2 \;+\; c_3x^3 \;+\; \dots$$

Of course, there is the question of convergence.

More generally, a power series "centered at a," is of the form:

$$c_0 \;+\; c_1(x-a) \;+\; c_2(x-a)^2 \;+\; c_3(x-a)^3 \;+\; \dots$$

Exercises.

1. Consider the function given by the power series

$$f(x) \;=\; \sum_{n=0}^{\infty} x^n \;=\; 1 + x + x^2 + x^3 + \dots$$

Notice that, for a fixed number x, this is a geometric series, with common ratio $r = x$. Therefore the series will converge if and only if $|x| < 1$. That means the domain of the function is $(-1, 1)$. In this exercise we will compute some values of $f(x)$.

(a) Find the value of $f(1/2) \;=\; \sum_{n=0}^{\infty} (1/2)^n$ using the formula for the sum of a geometric series.

(b) Make a table of x- and y-values for the function with $x = -9/10, -1/2, 0, 1/2, 9/10$. Plot the points to get a graph of $f(x)$ on its domain.

2. Consider the function defined by the power series

$$g(x) \;=\; \sum_{n=0}^{\infty} n!\, x^n \;=\; 1 + x + 2x^2 + 6x^3 + 24x^4 + \ldots$$

For what values of x will this series converge? (Hint: use the Ratio Test.)

What is the domain of $g(x)$?

3. For what values of x will $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

   What is the domain of this power series?

2. The Interval of Convergence

In general, the domain of a power series will be an interval, called the *interval of convergence.*

If the power series is centered at zero (i.e. it is of the form $\sum c_n x^n$), then the interval of convergence will be an interval centered around zero. The distance from zero to either endpoint is called the *radius of convergence.* The endpoints of the interval may or may not be included. Thus the interval of convergence for a power series centered at zero will be in one of the following forms:

$$(-\infty, \infty) \qquad (-R, R) \qquad [-R, R] \qquad (-R, R] \qquad [-R, R) \qquad \{0\}$$

The interval of a power series centered at $x = a$ will be one of the following forms:

$$(-\infty, \infty) \qquad (-R+a, R+a) \qquad [-R+a, R+a] \qquad (-R+a, R+a] \qquad [-R+a, R+a) \qquad \{a\}$$

We typically use the Ratio Test to determine the radius of convergence. Usually, further work is needed to determine convergence at the endpoints. See Examples 2, 3, 4, and 5 in the textbook.

Exercises

4. Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{5^n\, n^5}$.

   (a) Use the Ratio Test to show that the radius of convergence is $R = 5$.

(b) Since this power series is centered at zero and the radius of convergence is $R = 5$, the interval of convergence must be one of the following:

$$(-5, 5) \qquad [-5, 5] \qquad (-5, 5] \qquad [-5, 5)$$

i. What series do you get when you plug in $x = 5$? Does this series converge or not?

ii. What series do you get when you plug in $x = -5$? Does this series converge or not?

iii. Which of the four possibilities listed above is the actual interval of convergence for the power series?

5. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$.

(a) This power series is not centered at zero. What x-value is it centered around?

(b) Use the Ratio Test to determine the radius of convergence.

(c) What are the four possible intervals of convergence?

(d) Test the series at the endpoints to determine the interval of convergence.

6. The function J_1 defined by

$$J_1(x) \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

is called the Bessel function of order 1. (The Bessel function of order 0 is in Example 3.)

(a) What is the domain of $J_1(x)$?

(b) We can think of the series representation as a family of approximations. If $J_1(x) = \sum a_n(x)$, then the partial sum functions are

$$s_0(x) = a_0(x) \qquad s_1(x) = a_0(x) + a_1(x) \qquad s_2(x) = a_0(x) + a_1(x) + a_2(x) \qquad \text{etc.}$$

Use *Mathematica* to plot the first three partial sum functions on the same screen, and sketch your results below.