1 Overview

Main ideas:

1. unique representation theorem; coordinates of a vector relative to a particular basis
2. graphical interpretation of coordinates
3. change-of-coordinates matrix
4. coordinate mapping from abstract vsp to $\mathbb{R}^n$ is an isomorphism (one-to-one and onto linear transformation)

Examples in text:

1. coordinate vectors in $\mathbb{R}^2$
2. coordinates relative to standard basis for $\mathbb{R}^2$
3. crystalography
4. find coordinate vector in $\mathbb{R}^2$
5. coordinate mapping from $\mathbb{P}_3$ to $\mathbb{R}^4$
6. verify that certain polynomials in $\mathbb{P}_2$ are linearly dependent
7. determine whether a vector x in $\mathbb{R}^3$ is in the span of two given vectors v_1 and v_2, and write x in coordinates relative to v_1 and v_2

2 Discussion and Worked Examples

2.1 Coordinates Relative to a Basis

Having a finite basis for an abstract vector space allows us to treat the abstract vector space like $\mathbb{R}^n$. This is achieved by writing abstract vectors *in coordinates* relative to the basis. In this way, we can associate a finite list of real numbers (i.e. a vector in $\mathbb{R}^n$) to an abstract vector.

For example, a polynomial in $\mathbb{P}_3$ is uniquely determined by its coefficients. Recall that the standard basis for $\mathbb{P}_3$ is $\{1, x, x^2, x^3\}$. The *coordinates* of the polynomial $P(x) = 3 + 4x - 6x^2 + 11x^3$ relative to this basis are the coefficients $3, 4, -6$, and 11. The *coordinate vector* of $P(x)$ relative to this basis is the vector $(3, 4, -6, 11)$ in $\mathbb{R}^4$.

It is always possible to write an abstract vector in coordinates relative to a basis, if the abstract vector space *has* a *finite* basis. To see why this is true we need to do a little work.

Theorem If an abstract vector space V has a finite basis, then any vector in V can be written uniquely as a linear combination of basis vectors.

Proof Because the basis is a spanning set for V, any vector in V can be written as a linear combination of basis vectors. To show that this representation is unique, we need to use the linear independence of the basis.

Let $u_1, \ldots u_n$ be the basis vectors. Suppose that v can be written as a linear combination of basis vectors in two ways, i.e. suppose there are real numbers $c_1, \ldots, c_n$, and $d_1, \ldots, d_n$ such that

$$v = c_1u_1 + \ldots + c_nu_n \qquad \text{and} \qquad v = d_1u_1 + \ldots + d_nu_n$$

Subtracting, we get

$$0 = v - v = (c_1 - d_1)u_1 + \ldots + (c_n - d_n)u_n$$

The linear independence of the basis vectors implies that $c_1 = d_1, \ldots, c_n = d_n$, since the only way to write the zero vector as a linear combination of linearly independent vectors is to have all the weights be zero. $\square$

Note Given a finite, ordered basis $\mathcal{B}$ of an abstract vector space V, we now have a way to associate a unique vector in $\mathbb{R}^n$ (the coordinate vector with respect to $\mathcal{B}$) to a given vector in V. This association gives rise to a mapping from V to $\mathbb{R}^n$ called the *coordinate mapping.* Because of this tight association between vectors in V and vectors in $\mathbb{R}^n$, we will be able to answer questions about vectors in V by looking at the corresponding vectors in $\mathbb{R}^n$.

First we will show how to express a vector in $\mathbb{R}^2$ in coordinates relative to a basis for $\mathbb{R}^2$.

2.2 Examples in $\mathbb{R}^2$

Example Suppose $u_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $u_2 = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$, and $v = \left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$. Write v in coordinates relative to u_1 and u_2. What is the graphical interpretation of these coordinates?

First we write v as a linear combination of u_1 and u_2. In general, we can do this using row reduction of an augmented matrix, but in this case it is clear that $v = u_2 - 3u_1$. The coordinate vector for v relative to $\{u_1, u_2\}$ is $(-3, 1)$.

Note Note that the order of the coordinates matters! In order for the coordinate vector relative to a basis to be well-defined, we must fix an ordering of the basis.

Example Write $v = \left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$ in coordinates relative to the standard basis for $\mathbb{R}^2$. What is the graphical interpretation of these coordinates?

Since $v = -1e_1 + 1e_2$, the coordinate vector for v relative to the standard basis is $(-1, 1)$. This is, in fact, why $\{e_1, e_2\}$ is called the standard basis.

Notice that writing a vector b in $\mathbb{R}^n$ in coordinates relative to a basis for $\mathbb{R}^n$ comes down to solving a matrix-vector equation $Ax = b$, where A is the matrix whose columns are the given basis vectors. Since the columns of A are a basis for $\mathbb{R}^n$, A is invertible. Thus the inverse matrix A^{-1} is changes coordinates from the standard basis for $\mathbb{R}^n$ to a given basis, since $A^{-1}b$ yields the coordinate vector for b with respect to the columns of A. The linear transformation $b \mapsto A^{-1}b$ is the coordinate mapping mentioned above. On the other hand, the matrix A changes coordinates from a given basis to the standard basis for $\mathbb{R}^n$.

(In the text, the coordinate vector for a vector v with respect to a basis $\mathcal{B}$ is denoted $[\,v\,]_{\mathcal{B}}$. The matrix that changes coordinates from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$ is called the *change-of-coordinates matrix* and denoted $P_{\mathcal{B}}$. What we have observed is that $v = P_{\mathcal{B}}[\,v\,]_{\mathcal{B}}$.)

Example Find the matrix that changes coordinates from the standard basis for $\mathbb{R}^2$ to the basis $\mathcal{B} = \{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\5\end{smallmatrix}\right]\}$. Use this matrix to write $v = \left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]$ and $w = \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]$ in coordinates relative to $\mathcal{B}$.

The matrix that changes coordinates from the standard basis to $\mathcal{B}$ is

$$P_{\mathcal{B}}^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{5-2}\begin{bmatrix} 5 & -2 \\ -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 & -2 \\ -1 & 1 \end{bmatrix}$$

Thus the coordinate vector for v with respect to $\mathcal{B}$ is

$$[\,v\,]_{\mathcal{B}} = \frac{1}{3}\begin{bmatrix} 5 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 8/3 \\ -1/3 \end{bmatrix}$$

The coordinate vector for w with respect to $\mathcal{B}$ is

$$[\,w\,]_{\mathcal{B}} \;=\; \frac{1}{3}\begin{bmatrix} 5 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} \;=\; \begin{bmatrix} 4/3 \\ 1/3 \end{bmatrix}$$

Example Suppose that a vector v, in coordinates relative to $\mathcal{B}$, the basis for $\mathbb{R}^2$ in the previous example, is $[\,v\,]_{\mathcal{B}} \;=\; (-4, 5)$. What is v relative to the standard basis for $\mathbb{R}^2$?

In this case, we simply multiply the coordinate vector by the matrix whose columns are the basis elements:

$$v \;=\; P_{\mathcal{B}}\,[\,v\,]_{\mathcal{B}} \;=\; \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 5 \end{bmatrix} \;=\; \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

2.3 The Coordinate Mapping as an Isomorphism

As mentioned above, if a vector space V has a finite basis $\{u_1, \ldots, u_n\}$, we can associate vectors in V to vectors in $\mathbb{R}^n$ using the coordinate mapping. Observe that:

1. Given a vector $w = (c_1, \ldots, c_n)$ in $\mathbb{R}^n$, the vector $v = c_1u_1 \,+\, \ldots \,+\, c_nu_n$ in V will be mapped to w under the coordinate mapping.

2. Since two vectors in V that have the same coordinates relative to $\{u_1, \ldots, u_n\}$ are equal, this vector v in V, whose coordinate vector is w in $\mathbb{R}^n$, is unique.

Thus there is a one-to-one correspondence between vectors in V and vectors in $\mathbb{R}^n$.

Example Consider the coordinate vector $(1, -2, 3, -4)$ in $\mathbb{R}^4$. Clearly the the polynomial $P(x) = x - 2x + 3x^2 - 4x^3$ is the unique polynomial in $\mathbb{P}_3$ whose coordinates relative to the standard basis for $\mathbb{P}_3$ are $1, -2, 3$, and -4, respectively.

Observe that scaling P by a real number results in a polynomial whose coordinate vector (w.r.t. the std. basis for $\mathbb{P}_3$) is scaled by the same number.

$$5 \cdot P(x) \;=\; 5 - 10x + 15x^2 - 20x^3 \quad\longrightarrow\quad (5, -10, 15, -20) \;=\; 5(1, -2, 3, -4)$$

Now consider also $Q(x) = 2 + 3x - x^2 + x^3$ in $\mathbb{P}_3$, whose coordinate vector relative to the standard basis for $\mathbb{P}_3$ is $(2, 3, -1, 1)$. Notice that adding P and Q results in a polynomial whose coordinate vector is the sum of the coordinate vectors of P and Q.

$$P(x) \,+\, Q(x) \;=\; 3 + x + 2x^2 - 3x^3 \quad\longrightarrow\quad (3, 1, 2, -3) \;=\; (1+2,\; -2+3,\; 3-1,\; -4+1)$$

Generalizing this, we can see that the coordinate mapping is a linear transformation, since: (1) scaling a vector in V scales the weights occuring in the expression for v as a linear combination of basis vectors, which exactly corresponds to scaling the coordinate vector in $\mathbb{R}^n$, and (2) adding two vectors in V adds their weights with respect to the given basis, which exactly corresponds to adding coordinate vectors in $\mathbb{R}^n$.

Thus, the coordinate mapping from V to $\mathbb{R}^n$, given by $v \;\to\; [\,v\,]_{\mathcal{B}}$, where $\mathcal{B}$ is a basis of n vectors for V, is an *isomorphism*, i.e. a one-to-one and onto linear transformation. This means that V and $\mathbb{R}^n$ have exactly the same structure as vector spaces.

In particular, this means that we can again use row reduction to determine whether vectors (in a vector space with a *finite* basis) are linearly independent!

Example Determine whether the following polynomials are linearly independent in $\mathbb{P}_3$:

$$\begin{aligned} P(x) &= 1 - 2x^2 - x^3 \\ Q(x) &= x + 2x^3 \\ R(x) &= 1 + x - 2x^2 \end{aligned}$$

The coordinate vectors for $P(x)$, $Q(x)$, and $R(x)$ with respect to the standard basis for $\mathbb{P}_3$ are

$$\begin{aligned} P(x) &\to (1, 0, -2, -1) \\ Q(x) &\to (0, 1, 0, 2) \\ R(x) &\to (1, 1, -2, 0) \end{aligned}$$

These vectors in $\mathbb{R}^4$ are linearly independent if and only if the homogeneous matrix equation $Ax = 0$ has only the trivial solution, where A is the matrix whose columns are the given vectors.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a pivot in every column, the columns of A are linearly independent as vectors in $\mathbb{R}^4$. Thus the three polynomials are linearly independent in $\mathbb{P}_3$

How would we justify this rigorously? Well, recall that in your homework (4.3 #31-32), you showed that if $T : V \to W$ is a one-to-one linear transformation of vector spaces and $\{v_1, \ldots, v_p\}$ is a set of vectors in V, then $\{v_1, \ldots, v_p\}$ is linearly independent in V if and only if $\{T(v_1), \ldots, T(v_p)\}$ is linearly independent in W.

Note that if T is also onto $\{v_1, \ldots, v_p\}$ spans V if and only if $\{T(v_1), \ldots, T(v_p)\}$ spans W. It is worth thinking about why this is true.