1 Overview

Main ideas:

1. nonzero determinant is necessary and sufficient condition for invertibility
2. recursive definition of determinant of $n \times n$ matrix
3. cofactor expansion method for computing determinants
4. determinant of a triangular matrix is product of entries on the diagonal

Examples in text:

1. compute determinant of 3×3 matrix
2. compute determinant of 3×3 matrix using cofactor expansion
3. compute determinant of 5×5 matrix

2 Discussion and Worked Examples

Recall that a 2×2 matrix is invertible if and only if its determinant is nonzero. Similarly, the determinant of an $n \times n$ matrix is a certain number (and we will find a formula for this number soon) which can be used to determine whether the matrix is invertible.

For example, suppose A is an invertible matrix, whose (i, j)-th entry is denoted a_{ij}. Suppose that the first entry, a_{11} is nonzero and that we can reduce A to echelon form without switching rows. Then row reduction (see the textbook for the details) yields:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to \dots \to \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

where Δ is

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Since we are assuming a_{11} to be nonzero, the invertibility of A implies that Δ is nonzero. We will later see that the converse is true as well, and so A is invertible if and only if $\Delta \neq 0$. This Δ is the determinant of A.

We can rewrite the determinant in a form that is more memorable:

$$\Delta = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

where A_{ij} is the 2×2 submatrix of A obtained by crossing out the i-th row and the j-th column.

You might wonder what is so special about the first row. Why do we factor out a_{11}, a_{12}, etc instead of, say, a_{22}, a_{21}, and a_{23}, the elements of the second row? Or why not factor out a_{11}, a_{31} and a_{21}, the elements of the first column? Well, we certainly can. For example, proceeding down the second column:

$$\Delta = a_{22}(a_{11}a_{33} - a_{13}a_{31}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{32}(a_{13}a_{21} - a_{11}a_{23}) = -a_{12} \det A_{12} + a_{22} \det A_{22} - a_{32} \det A_{32}$$

Notice that in both of these formulas for the determinant, the signs alternate, but in the first formula they alternate $+, -, +$ and in the second formula they alternate $-, +, -$.

Example Compute the determinant of $A = \begin{bmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix}$.

We use the first formula, i.e. we proceed along the first row:

$$\det \begin{bmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix} = 0 \cdot \det \begin{bmatrix} -3 & 0 \\ 4 & 1 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 4 & -3 \\ 2 & 4 \end{bmatrix} = 0 - 5(4-0) + (16+6) = 2$$

Now, just for fun, proceeding down the second column, instead of across the first row:

$$\Delta = -5 \cdot \det \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} + (-3) \cdot \det \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} - 4 \cdot \det \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} = -5(4-0) + (-3)(0-2) - 4(0-4) = 2$$

Similarly, we can compute the determinant of a 4×4 matrix using determinants of suitable 3×3 submatrices. In general we can compute the determinant of an $n \times n$ matrix by computing determinants of suitable $(n-1) \times (n-1)$ submatrices. In particular, the first formula given above generalizes as follows:

$$\det A \overset{\text{DEF}}{=} a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n}$$

Again, we are proceeding across the first row. This formula is just fine, but we would like the flexibility of proceeding along other rows or columns. For example, if we wanted to compute the determinant of

$$\begin{bmatrix} 2 & 3 & 1 & 11 & -1 \\ 69 & 0 & -23 & 2 & -11 \\ -3 & 0 & 7 & -13 & 2 \\ 2 & 0 & 33 & 7 & -19 \\ 54 & 0 & 2 & -1 & 0 \end{bmatrix}$$

it would be very nice to be able to proceed down the second column instead of across the first row. The trick is to get the signs right. As in the 3×3 example, we need to start with a negative sign. In general, to determine which sign you start with, follow a checkerboard pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \\ + & - & & & \\ - & & \ddots & & \\ \vdots & & & & \end{bmatrix}$$

To state this formally, we look at the definition of the determinant again

$$\begin{aligned} \det A &\overset{\text{DEF}}{=} a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= a_{11} \left((-1)^{1+1} \det A_{11} \right) + a_{12} \left((-1)^{1+2} \det A_{12} \right) + \ldots + \left((-1)^{1+n} a_{1n} \det A_{1n} \right) \\ &= a_{11} C_{11} + a_{12} C_{12} + \ldots + a_{1n} C_{1n} \end{aligned}$$

where $C_{ij} = (-1)^{i+j} \det A_{ij}$ is the (i,j)-th cofactor of A. This formula is called the cofactor expansion across the first row of A. It can be modified to describe the analogous computation of the determinant by proceeding along any row or column.

Cofactor expansion across the i-th row:

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots + a_{in} C_{in}$$

Cofactor expansion down the j-th column:

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \ldots + a_{nj} C_{nj}$$

Example Compute the determinant of $A = \begin{bmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{bmatrix}$. (You should get -6.)