1 Overview

Main ideas:

1. In a vsp that has a basis with n elements, any set of more than n vectors is linearly dependent.
2. If a vsp V has a basis with n elements, then any basis for V has exactly n elements.
3. Formal definition of the dimension of a vector space.
4. A basis for a subspace can be expanded to a basis for the whole vector space. Thus the dimension of the subspace is less than or equal to the dimension of the whole vector space.
5. In an n-dimensional vector space, any linearly independent set of n vectors is a basis and any spanning set of n vectors is a basis.
6. The dimension of the null space of a matrix is the number of free variables in the corresponding homogeneous linear system, and the dimension of the column space is the number of pivot columns.

Examples in text:

1. dimension of $\mathbb{R}^n$ and $\mathbb{P}_n$
2. dimension of a subspace spanned by two vectors in $\mathbb{R}^3$
3. dimension of a subspace of $\mathbb{R}^4$ given in terms of four real parameters
4. classification of the subspaces of $\mathbb{R}^3$
5. dimensions of null space and column space of a matrix

2 Discussion and Worked Examples

2.1 Finite Bases for Vector Spaces

Recall from the previous section that, if a vector space V has a *finite basis*, then it is isomorphic to $\mathbb{R}^n$, where n is the number of elements in the basis. This means that, for all intents and purposes, V is just a "copy" of $\mathbb{R}^n$. To every abstract vector in V, we can assign a coordinate vector in $\mathbb{R}^n$. Adding and scaling of vectors in V corresponds exactly to adding and scaling the corresponding coordinate vectors in $\mathbb{R}^n$. In particular, this means that a set of vectors in V is linearly independent in V if and only if the corresponding coordinate vectors in $\mathbb{R}^n$ are linearly independent. This allows us to use row reduction to determine linear independence in an abstract vector space!

In this section, we continue exploring the nature of vector spaces with finite bases.

Let V be a vector space with a finite basis of n elements. Then

1. Any set of more than n vectors in V is linearly dependent.

   Suppose we have a set of m vectors in V with $m > n$. The collection of corresponding coordinate vectors is a set of m vectors in $\mathbb{R}^n$, with $m > n$, so must be linearly dependent. Thus the original set of vectors in V is also linearly dependent.

2. Any set of less than n vectors in V does not span V.

   Suppose we have a set of m vectors in V with $m < n$. The collection of corresponding coordinate vectors is a set of m vectors in $\mathbb{R}^n$, with $m < n$, so must not span $\mathbb{R}^n$. This means that there is a vector in $\mathbb{R}^n$ outside the span of the coordinate vectors. The corresponding vector in V is outside the span of the original set of m vectors in V.

$\star\star$ Thus, any basis for V must have exactly n elements. This is how we define the *dimension* of V.

3. Any linearly independent set of n vectors in V spans V and thus is a basis.

   The corresponding coordinate vectors form a linearly independent set of n vectors in $\mathbb{R}^n$, so are a basis for $\mathbb{R}^n$. Thus the original vectors are a basis for V.

4. Any set of n vectors that spans V is linearly independent and thus is a basis.

   Again, because the cooresponding coordinate vectors form a set of n vectors spanning $\mathbb{R}^n$, they are linearly independent. Thus the original vectors are linearly independent and thus a basis.

5. Any linearly independent set in V can be expanded to a basis for V by adding vectors not in the span, and any spanning set for V can be pared down to a basis for V by removing vectors that are linear combinations of the others, one at a time.

This is how we define the *dimension* of an abstract vector space. If an abstract vector space V has a finite basis with n elements, then any basis for V has n elements; the number n is intrinsic to the vector space and does not depend on the particular chosen basis, and it is called the *dimension* of V. A vector space with a finite basis is thus called a *finite dimensional* vector space, and a vector space that does *not* have a finite basis is called an infinite dimensional vector space.

For example, the vector space $\mathbb{P}_3$ is a finite dimensional vector space of dimension 4, since, as discussed last time, it is isomorphic to $\mathbb{R}^4$. The vector space $\mathbb{P}$ of *all* polynomials with real coefficients is infinite dimensional, because no finite set of polynomials spans the whole space.

Example Find the dimension of the subspace W below by finding a basis for W.

$$W = \left\{ \begin{bmatrix} 2a \\ -4b \\ -2a \end{bmatrix} : a, b, \text{ in } \mathbb{R} \right\}$$

Any vector w in W can be written as

$$w = \begin{bmatrix} 2a \\ -4b \\ -2a \end{bmatrix} = a \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + b \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} \qquad a, b, \text{ in } \mathbb{R}$$

Thus the vectors $(2, 0, -2)$ and $(0, -4, 0)$ span W. They are also visibly linearly independent. Thus they form a basis for W, and the dimension of W is 2.

Example Find the dimension of the subspace W below by finding a basis for W.

$$W = \left\{ \begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix} : p, q, \text{ in } \mathbb{R} \right\}$$

Any vector w in W can be written as

$$w = \begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix} = p \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix} + q \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad p, q, \text{ in } \mathbb{R}$$

Since the two vectors $(1, -1, 3, 1)$ and $(2, 0, -1, 1)$ are not scalar multiples of each other, they are linearly independent and thus form a basis for W, which thus has dimension 2.

In general, the dimension of a subspace is less than or equal to the dimension of the vector space in which it lies, because a basis for a subspace W of a vector space V cannot have more elements than the maximal number of linearly independent vectors in V. (Of course, this makes sense intuitively: one cannot fit a 3-dimensional space into the 2-dimensional plane without smashing down one of the dimensions.)

Example Determine the dimensions of the null space and column space of the matrix below:

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ -1 & 1 & 2 & 5 & 4 \\ 2 & -1 & -3 & 1 & 2 \\ 3 & 4 & 1 & 1 & -3 \end{bmatrix}$$

Row reduction yields

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ -1 & 1 & 2 & 5 & 4 \\ 2 & -1 & -3 & 1 & 2 \\ 3 & 4 & 1 & 1 & -3 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A_{\text{rref}}$$

The null space of A is the same as the null space of A_{rref}, which consists of vectors of the form:

$$v = t \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad s, t \text{ in } \mathbb{R}$$

Thus the null space is spanned by $(1, -1, 1, 0, 0)$ and $(0, 1, 0, -1, 1)$ which are visibly linearly independent. (Recall that the spanning set for the null space obtained from the reduced row echelon form is *always* linearly independent.) Thus the spanning set is in fact a basis, and the dimension of the null space is 2, which is the number of free variables.

While the column space for A is *not* the same as the column space of A_{rref}, the columns of A have the same dependency relations as the columns of A_{rref}. Since the third column of A_{rref} is a linear combination of the first two columns, and the fifth column of A_{rref} is a linear combination of the second and fourth column, the same is true for the corresponding columns of A. Thus removing the third and fifth columns of A leaves a linearly independent set that spans the column space of A_{rref}, i.e. a basis. So the column space of A is three-dimensional. Notice that we simply needed to count the number of pivot columns in A to determine this.

Pattern An $m \times n$ matrix with p pivot columns has $n - p$ free variables. The dimension of the column space is p, and the dimension of the null space is $n - p$.