1 Overview

Main ideas:

1. definition of eigenvector, eigenvalue, and eigenspace
2. finding eigenvectors, finding basis for eigenspace
3. eigenvalues of a triangular matrix are the diagonal entries
4. zero is an eigenvalue of a matrix if and only if the matrix is not invertible
5. linear independence of eigenvectors corresp to distinct eigenvalues
6. eigenvalues and difference equations

Examples in text:

1. concept of eigvector: matrix scales certain vectors
2. determine whether given vectors are eigvectors for given matrix
3. show that certain number is eigval and find corresp. eigvectors
4. find basis for eigsp corresp to given eigval
5. find eigvals of triangular matrices

2 Discussion and Worked Examples

2.1 Eigenvectors, Eigenvalues, Eigenspaces

Recall that an eigenvector for a matrix A is a vector that is simply scaled by the action of A. The scaling factor is called the eigenvalue. For example,

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So the vector $(1, 1)$ is an eigenvector for this matrix with eigenvalue 2.

Example Determine whether $(1, -1)$ is an eigenvector for the matrix above. If so, find the eigenvalue. How about $(1, 2)$? Eigenvalue?

Consider the vector $(1, -1)$.

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix} \qquad \Big(\text{not a scalar multiple of } \begin{bmatrix} 1 \\ -1 \end{bmatrix}\Big)$$

Thus $(1, -1)$ is not an eigenvector for the given matrix.

Now consider the vector $(1, 2)$

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So $(1, 2)$ is an eigenvector with eigenvalue 3.

Definition A nonzero vector v is an *eigenvector* for a matrix A if there is a number λ such that $Av = \lambda v$. (Note that v cannot be the zero vector, but λ may be zero.) The number λ is called an *eigenvalue* for A.

Clearly, given an eigenvector for a matrix, it is easy to find the corresponding eigenvalue. We would like to be able to determine the eigenvalues for a matrix without knowing the eigenvectors at the outset. By the definition above, a number λ is an eigenvalue if there is a nonzero vector v such that $Av = \lambda v$. But notice that

$$Av = \lambda v \iff Av - \lambda v = 0 \iff Av - \lambda \mathbb{I} v = 0 \iff (A - \lambda\mathbb{I})v = 0$$

Thus a number λ is an eigenvalue for A if and only if the homogeneous equation $(A - \lambda\mathbb{I})x = 0$ has a nontrivial solution! In fact, this is the textbook's definition of an eigenvalue.

Notice that the nonzero solution vectors to the equation $(A - \lambda\mathbb{I})x = 0$ are the eigenvectors corresponding to λ.

Example Determine whether -1 is an eigenvalue for the matrix $A = \begin{bmatrix} 2 & -10 & 5 \\ 0 & -1 & 0 \\ 0 & 4 & -3 \end{bmatrix}$. What about 3?

We need to determine whether null space of $(A - (-1)\mathbb{I})$ contains a nonzero vector.

$$(A - (-1)\mathbb{I}) = A + \mathbb{I} = \begin{bmatrix} 3 & -10 & 5 \\ 0 & 0 & 0 \\ 0 & 4 & -2 \end{bmatrix}$$

Notice that this matrix is not invertible, because its determinant is zero. Thus the null space must contain nonzero vectors, and -1 is an eigenvalue for A.

Next we look at $(A - 3\mathbb{I})$ and do a little row reduction:

$$(A - 3\mathbb{I}) = \begin{bmatrix} -1 & -10 & 5 \\ 0 & -4 & 0 \\ 0 & 4 & -6 \end{bmatrix} \to \begin{bmatrix} -1 & -10 & 5 \\ 0 & 4 & -6 \\ 0 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -10 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & -6 \end{bmatrix}$$

The echelon form shows that $(A - 3\mathbb{I})$ is invertible, thus the null space has no nonzero vectors and 3 is not an eigenvalue for A.

Example With A as above, find the eigenvectors corresponding to the eigenvalue $\lambda = -1$.

The eigenvectors corresponding to $\lambda = -1$ are the nonzero solutions to the equation $(A + \mathbb{I}) = 0$. We find the null space of this matrix using row reduction.

$$\begin{bmatrix} 3 & -10 & 5 \\ 0 & 0 & 0 \\ 0 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 3 & -10 & 5 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -10 & 5 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

A vector in the null space is of the form

$$x = \begin{bmatrix} 0 \\ (1/2)\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ (1/2) \\ 1 \end{bmatrix} \qquad \text{for } x_3 \text{ in } \mathbb{R}$$

Thus the eigenvectors corresponding to $\lambda = -1$ are the nonzero vectors in the span of $(0, 1, 2)$.

The set of solutions of the homogeneous equation $(A - \lambda\mathbb{I})x = 0$ is a subspace. (It is the null space of the matrix $(A - \lambda\mathbb{I})$.) It consists of all the eigenvectors corresponding to the eigenvalue λ together with the zero vector (which is not considered an eigenvector.) It is called the *eigenspace* of A corresponding to λ or the λ-eigenspace of A.

In the example above, the eigenspace for -1 is one-dimensional. In general, an eigenspace may be higher dimensional. To find a basis for the λ-eigenspace of a matrix simply find a basis for the null space of the matrix $(A - \lambda\mathbb{I})$.

2.2 Three Theorems

We have discussed how to

- check whether a given vector is an eigenvector for a given matrix and find the corresponding eigenvalue,
- check whether a given number is an eigenvalue for a given matrix and describe the corresponding eigenvectors,
- and find a basis for the eigenspace corresponding to a given eigenvalue,

but we do not yet have a way to find the eigenvalues of a matrix, without prior knowledge of the eigenvectors. We will discuss the general method for finding eigenvalues in Section 5.2, but in the meanwhile, we observe that:

> (Thm) The eigenvalues of a triangular matrix are the entries on the diagonal.

This is because the null space of $(A - \lambda\mathbb{I})$ will contain a nonzero vector if and only if one of the diagonal entries of $(A - \lambda\mathbb{I})$ is zero, i.e. λ equals one of the diagonal entries of A.

We have made a point of the fact that eigenvectors are, by definition, nonzero, but that an eigenvalue may be zero. What does an eigenvalue of zero tell us about the matrix?

> (Thm) A matrix A has zero as an eigenvalue if and only if $Ax = 0$ has nonzero solutions, i.e. A is not invertible.

We have found that the eigenvectors corresponding to a specific eigenvalue, together with the zero vector, form a subspace (the eigenspace), but how are eigenvectors corresponding to distinct eigenvalues related to each other? Could an eigenvector corresponding to one eigenvalue lie in the eigenspace of another eigenvalue?

Suppose λ and μ are two eigenvalues for a matrix A. Let v be a λ-eigenvector and w a μ-eigenvector. If v and w are linearly dependent, then w is a scalar multiple of v, since neither v nor w is zero. Let c be the scalar such that $cv = w$. Then

$$\mu w \;=\; Aw \;=\; A(cv) \;=\; c\,Av \;=\; c(\lambda v) \;=\; \lambda(cv) \;=\; \lambda w$$

But this implies that $\mu = \lambda$. Thus two eigenvectors corresponding to distinct eigenvalues are linearly independent. It turns out that this generalizes to the following theorem:

> (Thm) Eigenvectors corresponding to distinct eigenvalues are linearly independent.

See the textbook for a proof of this fact.