# 1 Overview

Main ideas in Section 5.2:

1. review of determinants
2. characteristic equation and char. polynomial of a matrix, algebraic multiplicity of an eigenvalue
3. similarity of matrices; similar matrices have the same eigenvalues
4. application to dynamical systems

Examples in Section 5.2:

1. finding eigenvalues of a $2 \times 2$ matrix
2. compute the determinant of a $3 \times 3$ matrix
3. find the characteristic equation of a $4 \times 4$ matrix
4. find the eigenvalues (with multiplicity) given the characteristic polynomial
5. analyze the long term behavior of a dynamical system

Main ideas in Section 5.3:

1. diagonalizable matrix; how to diagonalize a matrix; eigenvector basis of $\mathbb{R}^n$
2. sufficient criterion for diagonalizability of an $n \times n$ matrix: that it has $n$ distinct eigenvalues
3. general procedure for diagonalizing (a diagonalizable) matrix

Examples in Section 5.3:

1. computing powers of a $2 \times 2$ diagonal matrix
2. find a formula for the $k^{\text{th}}$ power of a given $2 \times 2$ matrix, given a diagonalization
3. diagonalize a $3 \times 3$ matrix
4. how diagonalization can fail
5. show that a given triangular $3 \times 3$ matrix is diagonalizable
6. diagonalize a $4 \times 4$ matrix that does *not* have distinct eigenvalues

# 2 Discussion and Worked Examples

## 2.1 Finding Eigenvalues

We have discussed how to find the eigenvalue corresponding to a given eigenvector (and how to find an eigenvector for a given eigenvalue.) Now we will discuss how to find an eigenvalue, without already having an eigenvector.

Recall that a number $\lambda$ is an eigenvalue for a (square) matrix $A$ if and only if the null space of $(A - \lambda\mathbb{I})$ contains more than just the zero vector. This is equivalent to saying that $(A - \lambda\mathbb{I})$ is not invertible. Thus we want to find all real numbers $\lambda$ such that $(A - \lambda\mathbb{I})$ is not invertible. The simplest way to do this is to think in terms of the determinant. Recall that a matrix is invertible if and only if its determinant is nonzero. Thus

$$\lambda \text{ is an eigenvalue for } A \iff \det(A - \lambda\mathbb{I}) = 0$$

**Example** Find the eigenvalues of the matrix $A = \begin{bmatrix} -1 & 5 \\ 10 & 4 \end{bmatrix}$.

We need to find all real numbers $\lambda$ such that $\det(A - \lambda\mathbb{I}) = 0$ .Computing,

$$\det(A - \lambda\mathbb{I}) = \det \begin{bmatrix} -1-\lambda & 5 \\ 10 & 4-\lambda \end{bmatrix} = (-1-\lambda)(4-\lambda) - (5)(10) = \lambda^2 - 3\lambda - 54 = (\lambda - 9)(\lambda + 6)$$

Thus the eigenvalues of $A$ are $\lambda = 9$ and $\lambda = -6$.

In general, the polynomial in $\lambda$ that results from computing the determinant of $\det(A - \lambda\mathbb{I})$ is called the *characteristic polynomial* of $A$, and the roots of the characteristic polynomial are the eigenvalues of $A$. If the characteristic polynomial has a double root, we say that the corresponding eigenvalue is of *multiplicity* two. In general, if the characteristic polynomial has a repeated root of order $n$, we say that the corresponding eigenvalue has multiplicity $n$.

## 2.2 Diagonalization

**Example** Consider the matrix $A = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix}$.

1. Find the eigenvalues and eigenvectors of $A$.
2. Let $P$ be the matrix with the eigenvectors for $A$ as its columns. Compute $AP$.
3. Compare the columns of $P$ and the columns of $AP$. What do you notice?
4. Let $D$ be the diagonal matrix whose entries are the eigenvalues of $A$, in the order corresponding to the order that the eigenvectors appear in the columns of $P$. Compute $PD$. What do you notice?

*Activity: Diagonalizing a matrix.*

In general, when it is possible to find eigenvectors of an $n \times n$ matrix $A$ forming a *basis* for $\mathbb{R}^n$, the matrix $P$, whose columns are these basis eigenvectors, is an invertible matrix, and $A = PDP^{-1}$, where $D$ is the diagonal matrix whose entries are the corresponding eigenvalues for $A$. This makes sense, because $P$ is the change-of-coordinates matrix from coordinates relative to the basis of eigenvectors to standard coordinates, and $P^{-1}$ changes coordinates from standard coordinates to coordinates relative to the basis of eigenvectors. Thus the product $PDP^{-1}$ changes to eigenvector coordinates, scales, and then changes back to standard coordinates!

The converse is also true, namely, if a matrix $A$ can be written in the form $A = PDP^{-1}$, where $D$ is a diagonal matrix, then there is a basis for $\mathbb{R}^n$ consisting of eigenvectors for $A$. In particular, the basis of eigenvectors can be taken to be the columns of $A$. To see why this is true, rewrite $A = PDP^{-1}$ as $AP = PD$, and notice that (1) $AP$ is the matrix whose columns are $Ap_1, \ldots, Ap_n$ (where $p_1, \ldots, p_n$ are the columns of $P$) and (2) $DP$ is the matrix whose columns are $\lambda_1 p_1, \ldots, \lambda_n p_n$ (where $\lambda_1, \ldots, \lambda_n$ are the entries on the diagonal of $D$). Thus $Ap_i = \lambda_i p_i$ for each column $p_i$ of $P$, i.e. the columns of $P$ (none of which can be the zero vector, since $P$ is invertible) are eigenvectors for $A$. Since $P$ is invertible, its columns form a basis for $\mathbb{R}^n$.

Note that the matrix $D$ occuring in the spectral decomposition of a matrix $A$ is not unique. Changing the order of the eigenvectors in the columns of $P$ would change the order of the eigenvalues appearing on the diagonal of $D$.

Recall that computing powers of matrices is laborious, but in certain cases it can be relatively simple. For example, computing powers of diagonal matrices is straightforward (because we simply compute the powers of the entries). The *spectral factorization* of a matrix $A = PDP^{-1}$, where $D$ is diagonal and $P$ is invertible, allows us to compute high powers of $A$ relatively easily, since $A^k = P\,D^k\,P^{-1}$.

## 2.3 Diagonalizability

We have shown that an $n \times n$ matrix $A$ is diagonalizable if and only if there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of $A$. We might wonder whether all matrices turn out to be diagonalizable or whether it is possible to have a matrix that does not have enough eigenvectors to provide a basis.

**Example** Consider the following four linear transformations of the plane:

| horiz & vert expn by 5 & 2 | dilation by 2 | horiz shear by 2 | rotation by 90° |
|---|---|---|---|
| $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ |

For each transformation, (a) find the characteristic polynomial, (b) find the eigenvalues and their multiplicities, (c) for each eigenvalue, determine the dimension of the corresponding eigenspace and find a basis, and (d) determine whether there is a basis for $\mathbb{R}^2$ consisting of eigenvectors for $A$.

*Activity: Obstructions to diagonalization*

Notice that two things can preclude the possibility of finding a basis for $\mathbb{R}^n$ consisting of eigenvectors for a given $n \times n$ matrix $A$:

1. The characteristic polynomial does not factor into linear factors (so there are less than $n$ roots of the characteristic polynomial, even when counting multiplicities).
2. The dimension for one (or more) of the eigenspaces is less than the algebraic multiplicity of the eigenvalue.

On the other hand, if the characteristic polynomial does factor into linear factors and the dimension of each eigenspace equals the algebraic multiplicity of the corresponding eigenvalue, then the sum of the dimensions of the eigenspaces is $n$, and since eigenvectors corresponding to distince eigenvalues are linearly independent, taking the union of the bases for the eigenspaces results in a basis for $\mathbb{R}^n$.

**Note:** The fact that the two obstructions described above are actually obstructions to diagonalization relies on the fact that the dimension of the $\lambda$-eigenspace is always less than or equal to the algebraic multiplicity of $\lambda$. We omit the proof of this fact, as does the textbook.

**Example** Determine whether $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ is diagonalizable.

We start by finding the characteristic polynomial

$$\det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (1-\lambda)\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda)^2$$

So $A$ has enough eigenvalues. The question is now whether the eigenspaces are sufficiently large. First we find a basis for the $\lambda = 1$ eigenspace:

$$\text{null} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \{x = (x_1, x_2, x_3) \,:\, x_2 = x_3 = 0\}$$

So a basis for the $\lambda = 1$ eigenspace is $\{(1,0,0)\}$. This has dimension 1, as expected. Next we look at the $\lambda = 2$ eigenspace:

$$\text{null} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \{x = (x_1, x_2, x_3) \,:\, x_1 = x_3 = 0\}$$

Thus a basis for the $\lambda = 2$ eigenspace is $\{(0,1,0)\}$. This is too small to span the rest of $\mathbb{R}^3$. Thus we cannot find a basis for $\mathbb{R}^3$ consisting of eigenvectors for $A$, and $A$ is not diagonalizable.

## 2.4 Similarity

**Definition** Suppose $A$ and $B$ are two (square) matrices such that $A = PBP^{-1}$ for some invertible matrix $P$. Then $A$ and $B$ are called *similar* matrices.

Thus a diagonalizable matrix is one that is similar to a diagonal matrix. Clearly a diagonalizable matrix $A$ has the same eigenvalues as a diagonal matrix, $D$, to which it is similar. It turns out that this is a special case of a more general fact, namely that any two similar matrices have the same eigenvalues.

**Claim** Similar matrices have the same characteristic polynomial, and thus the same eigenvalues with the same multiplicities.

*Proof:* Suppose $A$ and $B$ are similar matrices, i.e. there is an invertible matrix $P$ such that $A = PBP^{-1}$. Since we want to relate the characteristic polynomial of $A$ and the characteristic polynomial of $B$, we try to find a relationship between the matrices $B - \lambda\mathbb{I}$ and $A - \lambda\mathbb{I}$. We multiply $B - \lambda\mathbb{I}$ by $P$ and $P^{-1}$ on the left and right respectively. Then,

$$P\big(B-\lambda\mathbb{I}\big)P^{-1} \;=\; \big(PB-P(\lambda\mathbb{I})\big)P^{-1} \;=\; \big(PB-\lambda P\mathbb{I}\big)P^{-1} \;=\; \big(PB-\lambda P\big)P^{-1} \;=\; PBP^{-1}-\lambda(PP^{-1}) \;=\; A-\lambda\mathbb{I}$$

Thus, using the multiplicativity of the determinant,

$$\begin{aligned}\det(A-\lambda\mathbb{I}) \;&=\; \det\big(P(B-\lambda\mathbb{I})P^{-1}\big) \;=\; \det(P)\det(B-\lambda\mathbb{I})\det(P^{-1}) \;=\; \det(B-\lambda\mathbb{I})\det(P)\det(P^{-1}) \\ &=\; \det(B-\lambda\mathbb{I})\det(PP^{-1}) \;=\; \det(B-\lambda\mathbb{I})\end{aligned}$$

Thus the characteristic polynomial for $A$ is the same as the characteristic polynomial for $B$. Since the roots of the characteristic polynomials are the eigenvalues, with multiplicities, $A$ and $B$ have the same eigenvalues, with the same multiplicities. $\square$