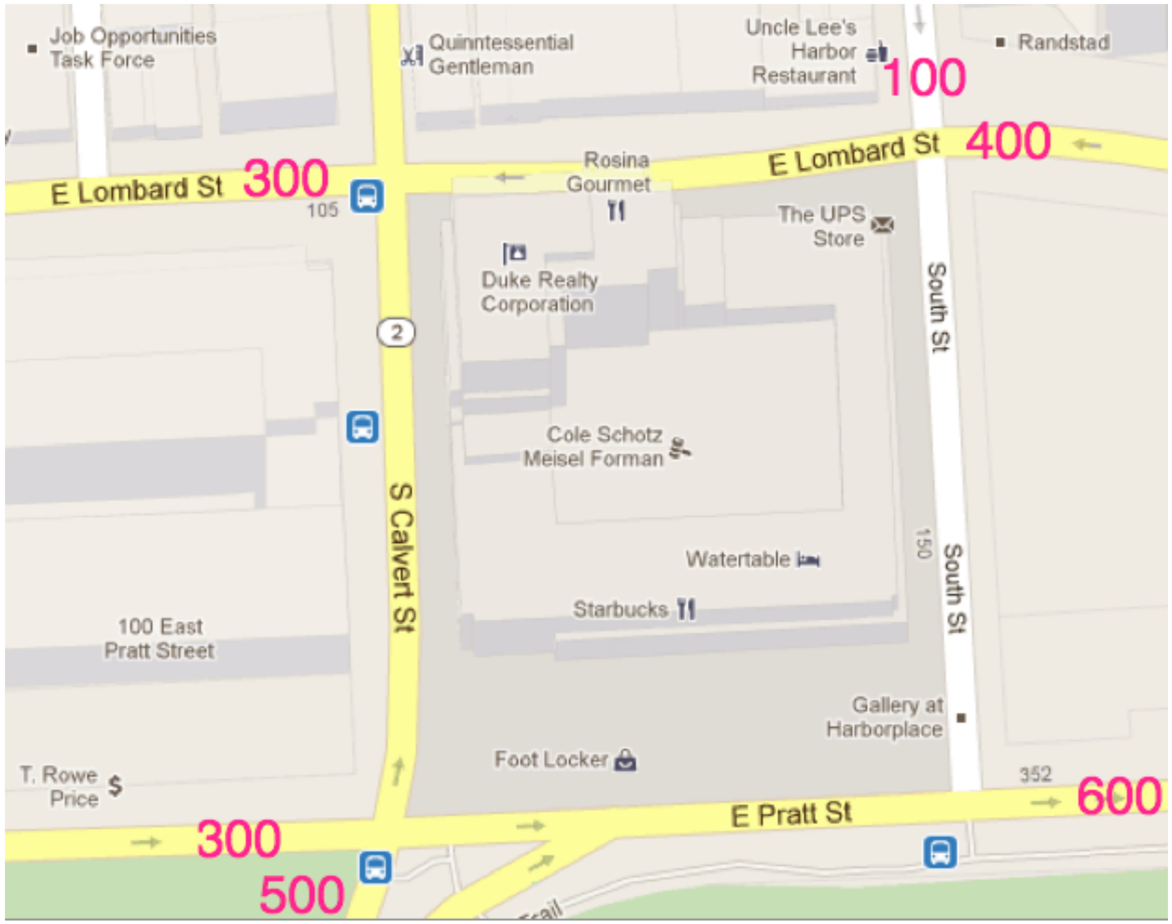
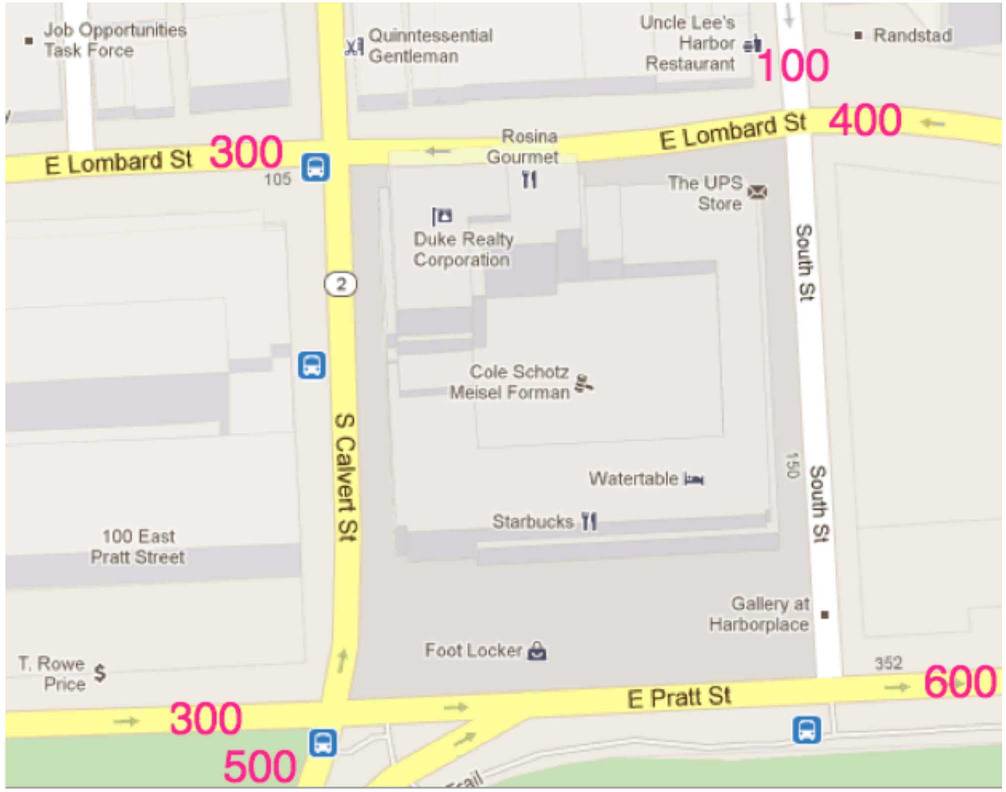
1. Traffic Flow Question

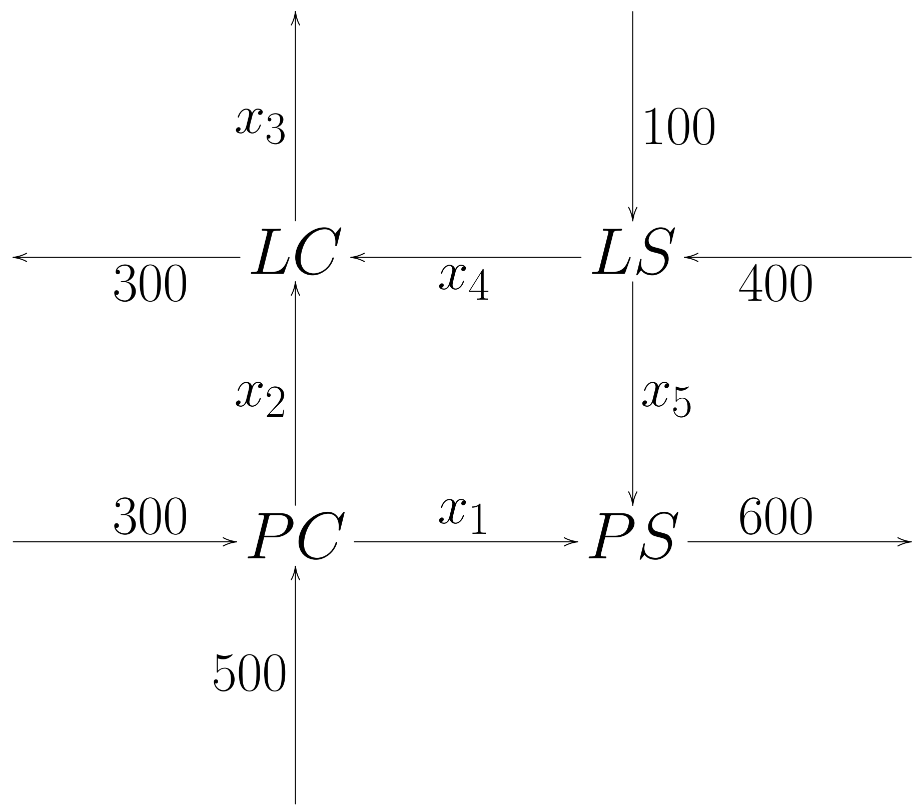


Where should I put my hot cocoa stand?

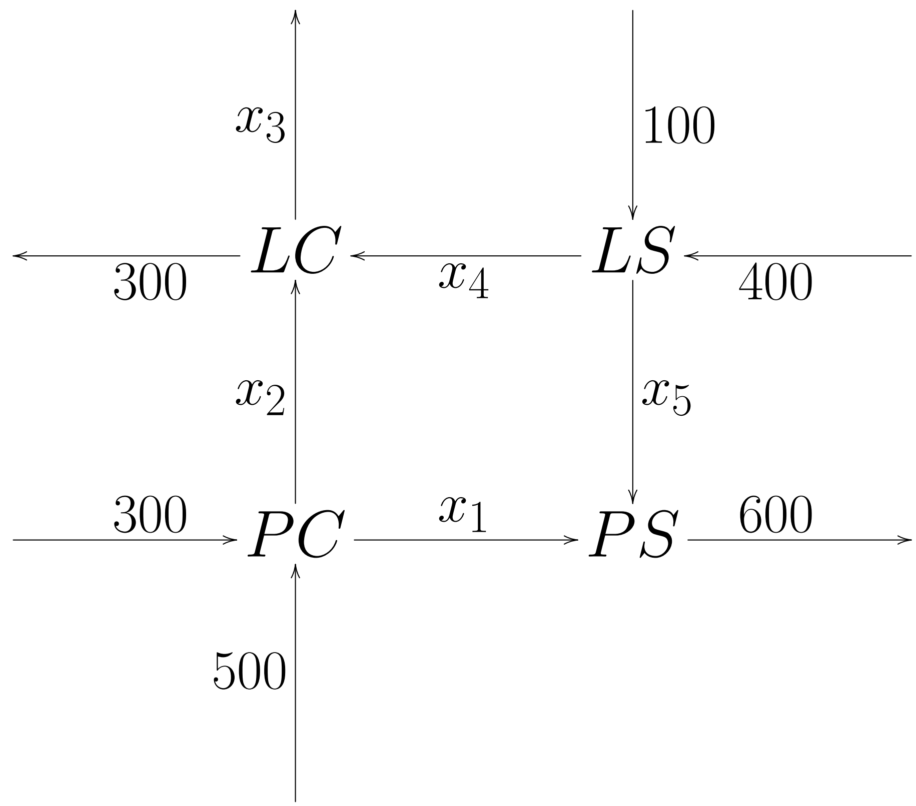


How many cars go north on Calvert St., after passing through the intersection between Calvert and Pratt? How many go east? What about the other intersections: Lombard and Calvert, South and Lombard, and Pratt and South?

Suppose we assign variables to each of these unknowns. What relationships can we find between the unknowns?



Key:



Thus we have the following system of (linear!) equations:

$$\begin{aligned} x_1 + x_2 &= 300 + 500 \\ x_2 + x_4 &= x_3 + 300 \\ 100 + 400 &= x_4 + x_5 \\ x_1 + x_5 &= 600 \end{aligned}$$

NB: 5 unknowns, but only 4 equations. Thus one is “free.”

Let $x_5 = A$, an undetermined constant and solve for the others in terms of A.

Simplfying:

$$\begin{aligned} x_1 + x_2 &= 800 \\ x_2 + x_4 &= x_3 + 300 \\ 500 &= x_4 + x_5 \\ x_1 + x_5 &= 600 \end{aligned}$$

Substituting $x_5 = A$ into the last equation and the third equation:

$$x_1 = 600 - A \qquad x_4 = 500 - A$$

Substituting x_1 in to the first equation and x_4 into the second equation:

$$x_2 = 200 + A \qquad x_3 = 400$$

Intersection	Flow (cars per day)
Pratt and Calvert	800
Lombard and Calvert	700
Lombard and South	500
Pratt and South	600

So, the intersection of Pratt and Calvert has the most traffic, and I should put my hot cocoa stand on that corner.

Note that we do not need to know the value of A in order to answer our question.

2. A Systematic Approach

Rewrite system. Work with associated **augmented matrix**.

$$
\begin{aligned}
x_1 + x_2 &= 800 \\
x_2 - x_3 + x_4 &= 300 \\
x_4 + x_5 &= 500 \\
x_1 + x_5 &= 600
\end{aligned}
\qquad
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 800 \\
0 & 1 & -1 & 1 & 0 & 300 \\
0 & 0 & 0 & 1 & 1 & 500 \\
1 & 0 & 0 & 0 & 1 & 600
\end{pmatrix}
$$

Elimination of variables on one hand; row operations on the other.

Add (-1) times the first equation to the fourth ($(-1)R_1 + R_4$)

$$\begin{aligned} x_1 + x_2 \qquad\qquad\qquad &= 800 \\ x_2 - x_3 + x_4 \qquad &= 300 \\ x_4 + x_5 &= 500 \\ -x_2 \qquad\qquad + x_5 &= -200 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & -1 & 0 & 0 & 1 & -200 \end{pmatrix}$$

Add the second equation to the fourth ($R_2 + R_4$).

$$\begin{aligned} x_1 + x_2 \qquad\qquad\qquad &= 800 \\ x_2 - x_3 + x_4 \qquad &= 300 \\ x_4 + x_5 &= 500 \\ -x_3 + x_4 + x_5 &= 100 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & -1 & 1 & 1 & 100 \end{pmatrix}$$

Switch the third and fourth equations ($R_3 \leftrightarrow R_4$).

$$\begin{aligned} x_1 + x_2 \qquad\qquad\qquad\qquad &= 800 \\ x_2 - x_3 + x_4 \qquad\quad &= 300 \\ - x_3 + x_4 + x_5 &= 100 \\ x_4 + x_5 &= 500 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & 1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{pmatrix}$$

Triangular form!

Q: Can we tell, at this stage, that there is at least one solution? That there are many? How?

Next work back upwards.

Add (-1) times fourth equation to third $((-1)R_4 + R_3)$

$$
\begin{aligned}
x_1 + x_2 &= 800 \\
x_2 - x_3 + x_4 &= 300 \\
- x_3 &= -400 \\
x_4 + x_5 &= 500
\end{aligned}
\qquad
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 800 \\
0 & 1 & -1 & 1 & 0 & 300 \\
0 & 0 & -1 & 0 & 0 & -400 \\
0 & 0 & 0 & 1 & 1 & 500
\end{pmatrix}
$$

.

Multiply third equation by (-1) $(R_3 \to -R_3)$

$$
\begin{aligned}
x_1 + x_2 &= 800 \\
x_2 - x_3 + x_4 &= 300 \\
x_3 &= 400 \\
x_4 + x_5 &= 500
\end{aligned}
\qquad
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 800 \\
0 & 1 & -1 & 0 & -1 & -200 \\
0 & 0 & 1 & 0 & 0 & 400 \\
0 & 0 & 0 & 1 & 1 & 500
\end{pmatrix}
$$

.

Add (-1) times fourth to second ($(-1)R_4 + R_2$)

$$\begin{aligned} x_1 + x_2 \qquad\qquad\qquad &= 800 \\ x_2 - x_3 \qquad - x_5 &= -200 \\ x_3 \qquad\qquad &= 400 \\ x_4 + x_5 &= 500 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 0 & -1 & -200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{pmatrix}$$

.

Add third equation to second ($R_3 + R_2$)

$$\begin{aligned} x_1 + x_2 \qquad\qquad\qquad &= 800 \\ x_2 \qquad\qquad - x_5 &= 200 \\ x_3 \qquad\qquad &= 400 \\ x_4 + x_5 &= 500 \end{aligned} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \end{pmatrix}$$

.

Add (-1) times second to first ($(-1)R_2 + R_1$):

$$\begin{array}{rcl} x_1 \qquad\qquad\qquad + x_5 &=& 600 \\ x_2 \qquad\qquad - x_5 &=& 200 \\ x_3 \qquad\qquad &=& 400 \\ x_4 + x_5 &=& 500 \end{array} \qquad \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 1 & 600 \\ 0 & \mathbf{1} & 0 & 0 & -1 & 200 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 400 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 500 \end{pmatrix}$$

.

Reduced row echelon form!

Basic variables: x_1, x_2, x_3, x_4 and free variable: x_5.

Parametric description of the solution set:

$x_1 = 600-A,\ \ x_2 = 200+A,\ \ x_3 = 400,\ \ x_4 = 500-A,\ \ x_5 = A$

where A is any (?) real number.

Solving linear systems

- substitution vs. elimination of variables vs. row reduction of augmented matrix
- basic variables and free variables, parametrizing solutions
- existence of solutions? uniqueness? (come back to this)

Elementary row operations:

- Multiply a row by a number: **row scaling** (e.g. $R_2 \to 2R_2$)
- Switch two rows: **row interchange** (e.g. $R_1 \leftrightarrow R_2$)
- Add (a multiple of) one row to another row, replacing the latter row with the sum: **row replacement** (e.g. $R_3 \to 2R_1 + R_3$)

Row Echelon Form

- Zero rows (if any) are at the bottom.
- Leading entry of each row to the right of leading entries of the rows above.
- Entries directly below leading entries are zeros.

$$\begin{bmatrix} \blacksquare & * & * & * & \dots & * \\ 0 & 0 & \blacksquare & * & \dots & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (\blacksquare \neq 0, \quad * = \text{anything})$$

Reduced Row Echelon Form

- Row echelon form and . . .

- Leading entries all ones.

- Entries directly above leading entries are zeros.

$$\begin{bmatrix} \mathbf{1} & * & 0 & 0 & * & * \\ 0 & 0 & \mathbf{1} & 0 & * & * \\ 0 & 0 & 0 & \mathbf{1} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (* = \text{anything})$$

3. Existence and Uniqueness of Solutions

Each matrix is the augmented matrix for a linear system. Write down the corresponding linear system. For each, determine if there is at least one solution. More than one?

$$A = \begin{bmatrix} 2 & 3 & 0 & -1 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 3 & 0 & -1 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 & 0 & -1 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & 0 & -1 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \longleftrightarrow$$

At least one solution? Many?

$$B = \begin{bmatrix} 2 & 3 & 0 & -1 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \qquad \longleftrightarrow$$

At least one solution? Many?

$$C = \begin{bmatrix} 2 & 3 & 0 & -1 & 1 \\ 0 & 1 & 1 & -3 & 7 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix} \qquad \longleftrightarrow$$

At least one solution? Many?

$$D = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \longleftrightarrow$$

At least one solution? Many?