1 Overview

Main ideas:

1. inner product (dot product) on $\mathbb{R}^n$, properties
2. length (norm) of a vector, unit vector, distance between two vectors
3. orthogonality, pythagorean theorem, angles in $\mathbb{R}^2$ and $\mathbb{R}^3$
4. the orthogonal complement to a subspace; theorem regarding complementarity of row space and null space

Examples in text:

1. compute $u \cdot v$ and $v \cdot u$ for u, v in $\mathbb{R}^3$
2. compute unit vector for a vector in $\mathbb{R}^4$
3. unit vector in a basis
4. distance between two vectors in $\mathbb{R}^3$
5. orthogonal complement to a plane through the origin in $\mathbb{R}^3$

2 Discussion and Worked Examples

2.1 Definitions and Basic Properties

The inner product (or scalar product or dot product) on $\mathbb{R}^n$ is the key to defining distance and perpendicularity in $\mathbb{R}^n$. An abstract vector space may or may not have an analogue of the dot product in $\mathbb{R}^n$. (But since every finite dimensional vector space is isomorphic to some $\mathbb{R}^n$, every finite dimensional vector space will have a dot product by "carrying over" the dot product on $\mathbb{R}^n$.)

Since vectors in $\mathbb{R}^n$ can be viewed as $1 \times n$ matrices, the *inner product*, denoted $\langle u, v\rangle$, or *dot product*, denoted $u \cdot v$, of two vectors in $\mathbb{R}^n$ can be defined as the matrix product $u^T v$. In coordinates:

$$\langle u, v\rangle \;=\; u \cdot v \;=\; u^T v \;=\; \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \;=\; u_1 v_1 \;+\; u_2 v_2 \;+\; \dots u_n v_n$$

The *length* (*norm*) of a vector is $\|u\| = \langle u, u\rangle^{1/2} \;=\; \sqrt{u \cdot u}$. The *unit vector* $\hat{u}$ pointing in the same direction as u is $\hat{u} = u/\|u\|$. The *distance* between two vectors u and v is $\|u - v\|$. (This is the usual distance formula: the square root of the sum of the squares of the differences of the coordinates.)

Two vectors are *orthogonal* if their dot product is zero. This coincides with the notion of perpendicularity of vectors in the plane. The Pythagorean Theorem, from a vector viewpoint, states that

$$u \text{ and } v \text{ are orthogonal} \iff \|u + v\|^2 \;=\; \|u\|^2 + \|v\|^2$$

The inner product on $\mathbb{R}^n$ can be characterized by the following three properties:

- positive definiteness: $\langle u, u\rangle \geq 0$, for all u in $\mathbb{R}^n$ and $\langle u, u\rangle = 0$ if and only if $u = 0$.
- symmetry: $\langle u, v\rangle = \langle v, u\rangle$, for all u, v in $\mathbb{R}^n$
- bilinearity:
  (i) $\langle cu, v\rangle = \langle u, cv\rangle = c\langle u, v\rangle$, for all u, v in $\mathbb{R}^n$ and all scalars c
  (ii) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ and $\langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle$ for all u, v, w in $\mathbb{R}^n$

2.2 The Orthogonal Complement to a Subspace

Suppose W is a plane through the origin in $\mathbb{R}^3$. Consider the set of vectors that are orthogonal to W, i.e. the line through the origin that is perpendicular to the plane. This set called the *orthogonal complement* of W, denoted $W^\perp$ (pronounced "W perp").

In general, if W is a subspace of $\mathbb{R}^n$, then

$$W^\perp \;=\; \{v \text{ in } \mathbb{R}^n : \langle v, w\rangle = 0 \text{ for all } w \text{ in } W\}$$

It is fairly straightforward (and part of your homework) to show that:

1. $W \cap W^\perp \;=\; \{0\}$
2. $W^\perp$ is a subspace of $\mathbb{R}^n$
3. If S is a spanning set for W, then v is in $W^\perp$ if and only if v is orthogonal to every vector in S.

Further, in $\mathbb{R}^n$ (and other finite dimensional vector spaces that have an inner product) the orthogonal complement of the orthogonal complement of a subspace is the original subspace, i.e.

$$\left(W^\perp\right)^\perp \;=\; W$$

for any subspace W of $\mathbb{R}^n$.

Example Describe the orthogonal complement of

$$W \;=\; \text{Span}\left\{ \begin{bmatrix} 4 \\ -4 \\ 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -5 \\ 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

It suffices to describe the set of vectors that are orthogonal to the vectors in the spanning set. So a vector $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ is in $W^\perp$ if and only if

$$\begin{array}{rcrcrcrcrcrcl} 4x_1 & - & 4x_2 & + & x_3 & - & x_4 & & & + & 3x_6 & = & 0 \\ 5x_1 & - & 5x_2 & + & x_3 & - & x_4 & + & x_5 & + & 2x_6 & = & 0 \\ 2x_1 & - & 2x_2 & & & & & + & x_5 & & & = & 0 \end{array}$$

i.e.

$$\begin{bmatrix} 4 & -4 & 1 & -1 & 0 & 3 \\ 5 & -5 & 1 & -1 & 1 & 2 \\ 2 & -2 & 0 & 0 & 1 & 0 \end{bmatrix} x \;=\; 0$$

Thus $W^\perp$ is the null space of the matrix whose rows are the vectors in the spanning set.

Theorem. *The orthogonal complement to the null space of a matrix A is the row space of A (i.e. the column space of A^T).*

$$\left(\text{null}A\right)^\perp \;=\; \text{col}(A^T) \;=\; \text{row}A$$

Proof. We will prove the following equivalent statement: $\text{null}A = \left(\text{row}A\right)^\perp$. A vector v is in null$A$ if and only if $Av = 0$, i.e. the dot products of the rows of A with v are all zero, i.e. v is orthogonal to the rows of A, considered as vectors. Since the rows of A are a spanning set for the row space of A, this is equivalent to v being in the orthogonal complement of row$A$. □