1 Overview

Main ideas:

1. definitions of linear independence, linear dependence, dependence relation, basis
2. characterization of linearly dependent set using linear combinations (thm 4)
3. paring down a spanning set to find a basis (thm 5)
4. bases for the null space and column space of a matrix (thm 6)
5. basis is minimal spanning set and maximal linearly independent subset

Examples in text:

1. linearly dependent set in $\mathbb{P}$
2. linearly independent and dependent sets in $C[0,1]$, the vsp of continuous functions on $[0,1]$
3. columns of an invertible matrix
4. standard basis for $\mathbb{R}^n$
5. determine whether three vectors in $\mathbb{R}^3$ are a basis for $\mathbb{R}^3$
6. standard basis for $\mathbb{P}_n$
7. find a basis for the subspace spanned by a given set
8. find a basis for the column space of a matrix in rref
9. find a basis for the column space of a matrix *not* in rref
10. enlarging a linearly independent set to a basis and beyond

2 Discussion and Worked Examples

2.1 Linear Independent Sets and Spanning Sets in $\mathbb{R}^n$

(Warm-up: For each set of vectors in $\mathbb{R}^2$, determine whether the set is linearly independent and describe the span.)

Two Observations (1) Two vectors in $\mathbb{R}^2$ that do not lie on the same line span the whole plane. (2) A set of three or more vectors in $\mathbb{R}^2$ is always linearly dependent.

Let's take some time to recap the arguments for why these facts are true. Let $\{v_1, \ldots, v_n\}$ be a set of n vectors in $\mathbb{R}^m$, and let A be the matrix whose columns are the vectors $v_1, \ldots, v_n$. Then

$$\{v_1, \ldots, v_n\} \text{ spans } \mathbb{R}^m \iff Ax = b \text{ is consistent for all } b \text{ in } \mathbb{R}^m \iff A \text{ has a pivot in every row}$$

$$\{v_1, \ldots, v_n\} \text{ is lin. indep.} \iff Ax = 0 \text{ has only the trivial solution} \iff A \text{ has a pivot in every col.}$$

When $n = m$, A has the same number of rows and columns, so if it has a pivot in every row, it must have a pivot in every column, and vice versa. Thus

$$\begin{array}{ccccc}
\{v_1, \dots, v_n\} \text{ spans } \mathbb{R}^n & \iff & Ax = b \text{ is consistent for all } b \text{ in } \mathbb{R}^n & \iff & A \text{ has a pivot in every row} \\
& & & & \Updownarrow\ (n = m) \\
\{v_1, \dots, v_n\} \text{ is lin. indep.} & \iff & Ax = 0 \text{ has only the trivial solution} & \iff & A \text{ has a pivot in every col.}
\end{array}$$

Thus a set of n vectors in $\mathbb{R}^n$ spans $\mathbb{R}^n$ if and only if it is linearly independent.

Also, notice that any set of n vectors in $\mathbb{R}^m$ (with $n > m$) is linearly dependent, because if A has more columns than rows, it cannot have a pivot in every column.

Further, a set of n vectors in $\mathbb{R}^m$ (with $n < m$) cannot span $\mathbb{R}^m$, because if A has more rows than columns, it cannot have a pivot in every row.

A linearly independent spanning set for $\mathbb{R}^n$ is called a *basis* for $\mathbb{R}^n$. What we have observed in the plane and proven in $\mathbb{R}^n$ is that:

- Any linearly independent set of exactly n vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.
- Any spanning set for $\mathbb{R}^n$ that has exactly n vectors is a basis for $\mathbb{R}^n$.
- A basis for $\mathbb{R}^n$ cannot have more than n vectors (or it would be linearly dependent) and cannot have less than n vectors (or it would not span). Thus a basis for $\mathbb{R}^n$ has exactly n vectors.

Note The columns of an $n \times n$ invertible matrix form a basis for $\mathbb{R}^n$, by the Invertible Matrix Theorem.

Note Given a linearly independent set in $\mathbb{R}^n$ that does not span $\mathbb{R}^n$, we can add new vectors to make a basis. Similarly, given a linearly dependent spanning set in $\mathbb{R}^n$, we can remove vectors to make a basis.

Example For each of the sets of vectors in $\mathbb{R}^2$ above, either add or remove vectors to make a basis.

In order to define a basis for an abstract vector space we need an abstract notion of linear independence.

2.2 Linear Independent Sets and Bases in an Abstract Vector Space

Definition Let $\{v_1, v_2, \dots, v_p\}$ be a subset of a vector space V. The set is linearly dependent if there are weights $c_1, c_2, \dots, c_p$, some of which may be zero, but not all, such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0$$

Otherwise the set is linearly independent: the only set of weights $c_1, c_2, \dots, c_p$ satisfying

$$c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0$$

is the set of all zero weights $c_1 = c_2 = \cdots = c_p = 0$.

A set consisting of just one vector is linearly dependent if and only if it is the set consisting of the zero vector. A set of more than one vector is linearly dependent if and only if one of the vectors can be written as a linear combination of the others. (This is Theorem 4 in the text.)

Note Recall that in $\mathbb{R}^n$ a set of vectors is linearly independent if and only if the homogeneous equation $Ax = 0$ has only the trivial solution, where A is the matrix whose columns consist of the vectors in the specified set of vectors. However, for an abstract vector space, we may not be able to write vectors as columns of a matrix.

Example Consider the following polynomials in $\mathbb{P}$ and determine whether they are linearly dependent or independent:

$$P(x) \;=\; 2 \qquad Q(x) \;=\; x^2 \qquad R(x) \;=\; x^2 + 1$$

Example The functions $\sin(x)$ and $\cos(x)$ are linearly independent in the vsp of functions on $\mathbb{R}$, but the functions $\sin(2x)$ and $\sin(x)\cos(x)$ are linearly dependent, since

$$\sin(2x) \;=\; 2\sin(x)\cos(x)$$

Definition A linearly independent spanning set of a vector space V is called a basis for V. (The plural of "basis" is "bases.")

The *standard basis* for $\mathbb{R}^n$ is $\{e_1, e_2, \ldots, e_n\}$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The *standard basis* for $\mathbb{P}_n$ is $\{1, x, x^2, \ldots, x^n\}$. Certainly any polynomial with real coefficients of degree $d \leq n$ can be written as a linear combination of these monomials. The fact that they are linearly independent is also clear, because all of the coefficients of the zero polynomial are zero.

As in $\mathbb{R}^n$, a spanning set for a vector space V that is linearly dependent can be "pared down" to a basis for V, by removing vectors that can be written as linear combinations of previous vectors in the spanning set, and a linearly independent set that does not span V can be "beefed up" to a basis for V by adding vectors to the set that are linearly independent.

$$\begin{tikzcd} \begin{array}{c}\text{lin. indep. set that}\\ \text{does not span}\end{array} \arrow[r, bend left, "\text{add vectors}"] & \text{BASIS} \arrow[r, bend left, "\text{add vectors}"] \arrow[l, bend left, "\text{remove vectors}"] & \begin{array}{c}\text{lin. dept}\\ \text{spanning set}\end{array} \arrow[l, bend left, "\text{remove vectors}"] \end{tikzcd}$$

In this sense, a basis is a *minimal* spanning set and a *maximal* linearly independent set.

2.3 Bases for the Null Space and the Column Space of a Matrix

Recall that row reduction can be described in terms of multiplication by elementary matrices.

$$A \;\to\; E_1 A \;\to\; \dots \;\to\; E_r \dots E_1\, A \;\text{ (rref)}$$

Let A_{rref} denote the reduced row echelon form of A. If $Ax = 0$, then $A_{\text{rref}}\, x \;=\; E_r \dots E_1(Ax) = 0$. On the other hand, if $A_{\text{rref}}\, x = 0$, then $(E_r \dots E_1)Ax = 0$ and thus $Ax = (E_r \dots E_1)^{-1}(0) = 0$, since elementary matrices are invertible. Thus

$$Ax \;=\; 0 \quad\Longleftrightarrow\quad A_{\text{rref}}\, x \;=\; 0$$

This key observation allows us to determine bases for the null space and the column space of a matrix from the reduced row echelon form of the matrix.

Example Find bases for the null spaces and column spaces of the matrices A and B, which are given below along with their reduced row echelon forms.

$$A \;=\; \begin{pmatrix} 1 & 2 & 1 & 3 & 1 \\ -1 & 1 & 2 & 5 & 4 \\ 2 & -1 & -3 & 1 & 2 \\ 3 & 4 & 1 & 1 & -3 \end{pmatrix} \qquad A_{\text{rref}} \;=\; \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B \;=\; \begin{pmatrix} 1 & 2 & 1 & 3 & 1 \\ -1 & 1 & 2 & 5 & 4 \\ 2 & -1 & -3 & 1 & 2 \\ 3 & 4 & 1 & 1 & 2 \end{pmatrix} \qquad B_{\text{rref}} \;=\; \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $Ax = 0$ if and only if $A_{\text{rref}}\, x = 0$, the null space of a matrix is the *same* as the null space of its reduced row echelon form. Recall that an explicit description for the null space of a matrix is obtained by writing the solution set for the associated homogeneous equation in parametric vector form. A vector x is in the null space of A if and only if it is of the form

$$x \;=\; \begin{bmatrix} x_3 \\ -x_3 + x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} \;=\; x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \;+\; x_5 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Thus the null space is spanned by the vectors $v = (1, -1, 1, 0, 0)$ and $u = (0, 1, 0, -1, 1)$. Notice that these two vectors are linearly independent, since, if $cv + du = 0$, then

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \;=\; c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \;+\; d \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} c \\ -c + d \\ c \\ -d \\ d \end{bmatrix}$$

and, in particular, this implies that $c = 0$ and $d = 0$. Recall that whenever we obtain a spanning set for the null space of a matrix from the reduced row echelon form of the matrix, the spanning set will be linearly independent. In other words, this procedure always yields a basis for the null space of the original matrix.

Similarly, to find a basis for the null space of B, we write the solution set for $Bx = 0$ in parametric vector form. A vector x is a solution to $Bx = 0$ if and only if

$$x = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \\ 0 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus a basis for the null space of B is $\{(1, -1, 1, 0, 0)\}$.

Recall that the column space of A is the span of the columns of A. Thus, for a given matrix A, we already have a spanning set for the column space of A, namely the columns of A. However, the columns may be linearly dependent, in which case we need to remove some of the vectors to obtain a linearly independent spanning set, i.e. a basis.

Recall that we can interpret the matrix-vector product Ax as a linear combination of the columns of A, where the entries of x are the weights in the linear combination. The columns of A are linearly independent if and only if the only way to weight the columns of A to get the linear combination to equal zero is to have all the weights be zero, i.e. the only solution to $Ax = 0$ is the zero vector $x = 0$. We can tell from the reduced row echelon form of A whether there are nontrivial solutions to $Ax = 0$, since $Ax = 0$ if and only if $A_{\text{rref}}\, x = 0$.

Looking at the matrices A and B above, it is clear that $Ax = 0$ and $Bx = 0$ both have nontrivial solutions.

If the columns of a matrix A are linearly dependent, each nonzero vector x such that $Ax = 0$ gives a dependency relation among the columns of A. Since $Ax = 0$ if and only if $A_{\text{rref}}\, x = 0$, the dependency relations among the columns of A are the same as the dependency relations among the columns of A_{rref}. For example, if the third column of A_{rref} is a linear combination of the first and second columns of A_{rref}, then the third column of A is a linear combination of the first and second columns of A.

Look at the matrix A_{rref} above. Notice that the third column is a linear combination of the first two columns and the fifth column is a linear combination of the second and fourth columns. Thus the third column and the fifth columns are "redundant" in the following sense: the span of columns of A_{rref} is the same as the span of the first, second, and fourth columns of A_{rref}. Since the first, second, and fourth columns are linearly independent, and they span the column space of A_{rref}, they are a basis for the column space of A_{rref}.

Since the dependency relations for the columns of A are the same as the dependency relations of the columns of A_{rref}, the third column of A is a linear combination of the first and second columns and the fifth column of A is a linear combination of the second and fourth columns of A. Thus the first, second, and fourth columns of A,

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}$$

form a basis for the column space of A.

Similarly, since the third column of B_{rref} is a linear combination of the first and second columns of B_{rref}, we may remove it from the spanning set without changing the span. The remaining four columns are linearly independent, so form a basis for the column space of B_{rref}. Thus the first, second, fourth, and fifth columns of B,

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

form a basis for the column space of B.

Summary:

1. To find a basis for the null space of a matrix A:

   Find A_{rref}, write the solution set to $A_{\text{rref}}\, x = 0$ in parametric vector form, using the free variables as the parameters. The spanning set for the null space obtained in this way is a basis.

2. To find a basis for the column space of a matrix A:

   The columns of A are a spanning set for the column space of A. To determine which columns to remove in order to ensure linear independence, find A_{rref}. If any columns of A_{rref} are linear combinations of previous columns, remove the corresponding columns of A from the spanning set. The resulting spanning set will be a basis.