1 Overview

Change of perspective: a matrix defines a function whose inputs and outputs are vectors. We will be able to give a geometric interpretation to the matrix-vector equation $Av = b$.

Main ideas:

1. transformation perspective (terms: function/transformation/mapping, domain, codomain, range)
2. for $m \times n$ matrix A: domain is $\mathbb{R}^n$, codomain $\mathbb{R}^m$, and range is span of columns of A
3. existence and uniqueness questions
4. geometric viewpoint
5. definition: linear transformation (vs matrix transformation), simple properties

Examples in text:

1. matrix transformation (image of vector, preimage of vector, range of transformation)
2. projection matrix
3. shear tranformation
4. contraction/dilation
5. rotation
6. application

2 Example

Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$ where

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}, \quad \text{and let } u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad b = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}, \quad \text{and } c = \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}.$$

(a) Find $T(u)$, the image of u under the transformation T.

$$T(u) = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

(b) Find an v in $\mathbb{R}^2$ whose image under T is b.

Using row reduction

$$\begin{bmatrix} 1 & -1 & -5 \\ 0 & 1 & 3 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & -1 & -5 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{bmatrix} \xrightarrow{(1/3)R_3} \begin{bmatrix} 1 & -1 & -5 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{-R_2 + R_3} \begin{bmatrix} 1 & -1 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $v = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(c) Is there more than one v whose image under T is b?

No, the linear system corresponding to the matrix-vector equation $Av = b$ has a unique solution.

(d) Determine if c is in the range of the transformation T.

Using row reduction

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -4 \\ 2 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -4 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{-4R_2 + R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 17 \end{bmatrix}$$

This system is inconsistent, so c is not in the range of T.