1 Overview

Main ideas:

1. vector perspective of homogeneous system of equations
2. linear dependence and independence of vectors
3. sets of one and two vectors
4. linear dependence and linear combinations
5. easy check: dimension counting or presence of zero vector

Examples in text:

1. determine whether a set of vectors is linearly independent, and find a dependence relation if it is not
2. determine whether the columns of a matrix are linearly independent
3. determine by inspection whether a pair of vectors is linearly independent
4. describe the span of a pair of vectors and explain connection between span and linear dependence
5. determine by inspection whether three vectors in $\mathbb{R}^2$ are linearly independent
6. determine by inspection whether the given set is linearly independent

2 Discussion

Recall discussion of linear combinations and span: in one example, two vectors in $\mathbb{R}^2$ spanned the whole plane and, in the other example, two vectors spanned a line (and each of them could have spanned the line by itself.) In the first example, the two vectors are *linearly independent* and in the second example the two vectors are *linearly dependent.*

In $\mathbb{R}^2$ two vectors are linearly dependent if and only if they lie on the same line through the origin, i.e. they are scalar multiples of each other. A set of three vectors in $\mathbb{R}^3$ is linearly independent if they do not all lie on the same plane.

A set of vectors $\{v_1, \ldots, v_n\}$ is linearly independent if:

- v_1 is nonzero
- v_2 is not in the span of v_1
- v_3 is not in the span of $\{v_1, v_2\}$
- v_4 is not in the span of $\{v_1, v_2, v_3\}$

$$\vdots$$

- v_n is not in the span of $\{v_1, \ldots, v_{n-1}\}$

i.e., none of the vectors can be written as a linear combination of the other vectors.

This description of linear independence is not usually taken as the definition, but it is an intuitive notion which helps us get off the ground, so to speak.

Example Determine whether the following sets of vectors are linearly independent:

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \end{bmatrix} \right\}$

Note Any set of vectors containing the zero vector is linearly dependent, because zero lies in the span of any set of vectors.

While we can determine whether certain sets of vectors are linearly independent "by inspection," as above, the formal definition is useful for determining linear independence in cases that are not as clear.

Formal Definition of Linear Dependence/Independence

A set $\{v_1, \ldots, v_n\}$ of vectors is linearly dependent if there exist weights $c_1, \ldots, c_n$, not all of which are zero, such that

$$c_1 v_1 + \ldots + c_n v_n = 0$$

A set of vectors that is not linearly dependent is linearly independent, namely a set $\{v_1, \ldots, v_n\}$ of vectors is linearly independent if the vector equation

$$c_1 v_1 + \ldots + c_n v_n = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.

Matrix-vector Equation Viewpoint Changing from the vector equation viewpoint to the matrix-vector equation viewpoint, the columns of a matrix A are linearly dependent if and only if the homogeneous matrix-vector equation $Av = 0$ has a non-trivial solution. (The v_i in the vector equation formulation of linear independence become the columns of A, and the weights c_i are the entries of the solution vector v.) This means that we can use our row reduction algorithm to determine linear independence!

Example Determine whether the set of vectors is linearly independent:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \end{bmatrix} \right\}$$

By the observation above, this is equivalent to determining whether the homogeneous linear system $Av = 0$ has a nontrivial solution, where A is the matrix whose columns are the given vectors:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \\ 1 & 0 & 3 \end{bmatrix}$$

Row reduction of the augmented matrix yields

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix} \longrightarrow \dots \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x_1 = -3x_3$, $x_2 = x_3$, and x_3 is free. The parametric vector form of the solution set is thus

$$v = t \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \quad t \text{ in } \mathbb{R}$$

In particular there are nontrivial solutions. Further, this gives us the weights in the dependence relation:

$$-3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This makes it clear that the third vector is a linear combination of the first two, because the third vector is equal to three times the first vector minus the second vector.

Note Each nonzero value of t would yield a set of weights. The corresponding vector equation would be equivalent to the one above, because choosing another value for t amounts to multiplying both sides of the vector equation by a constant.

Note This matrix-vector perspective also makes it clear that any set of more than n vectors in $\mathbb{R}^n$ must be linearly dependent, because if a matrix A has more columns than rows, then the equation $Av = 0$ must have nontrivial solutions.