1 Overview

Main ideas:

1. Linear systems: linear equation, coefficients, system of linear equations (linear system), solution of linear system, solution set, equivalence of linear systems, consistent and inconsistent linear systems (algebraic and geometric interpretation)
2. Matrix notation: matrix, coefficient matrix and augmented matrix of a linear system, size of matrix (rows by columns)
3. Solving a linear system by elimination and the corresponding actions on augmented matrix (elementary row operations)
4. Existence and uniqueness of solutions

Examples in text:

1. Solving a (consistent) linear system (with a unique solution)
2. Demonstrating the consistency of a linear system (by finding a solution)
3. Demonstrating the inconsistency of a linear system (by arriving at a contradiction)

2 Worked Examples

Example 1 Consider the following linear system:

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & 6y & + & 3z & = & 0 \\ -12x & + & 6y & + & 2z & = & -40 \end{array}$$

The corresponding augmented matrix is:

$$\begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ -12 & 6 & 2 & -40 \end{pmatrix}$$

Notice what happens to the augmented matrix as we solve the linear system by elimination.

First multiply the top equation by 3, and add it to the bottom equation:

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & 6y & + & 3z & = & 0 \\ & & & & 5z & = & 20 \end{array} \qquad \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 5 & 20 \end{pmatrix} \qquad \text{(Triangular!)}$$

Divide the bottom equation by 5 to find z:

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & 6y & + & 3z & = & 0 \\ & & & & z & = & 4 \end{array} \qquad \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Now we want to find y. Multiply the bottom equation by -3, and add it to second equation.

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & 6y & & & = & -12 \\ & & & & z & = & 4 \end{array} \qquad \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 0 & -12 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Divide second equation by 6 to solve for y:

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & y & & & = & -2 \\ & & & & z & = & 4 \end{array} \qquad \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Now we want to find x. Multiply the bottom equation by -1, and add it to the top equation.

$$\begin{array}{rcrcrcr} 4x & - & 2y & & & = & 16 \\ & & y & & & = & -2 \\ & & & & z & = & 4 \end{array} \qquad \begin{pmatrix} 4 & -2 & 0 & 16 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Multiply the second equation by 2, and add it to the first:

$$\begin{array}{rcrcrcr} 4x & & & & & = & 12 \\ & & y & & & = & -2 \\ & & & & z & = & 4 \end{array} \qquad \begin{pmatrix} 4 & 0 & 0 & 12 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Divid the first equation by 4 to find x:

$$\begin{array}{rcrcrcr} x & & & & & = & 3 \\ & & y & & & = & -2 \\ & & & & z & = & 4 \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

We can see that this system has a unique solution. Notice the form of the augmented matrix:

$$\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$$

where the stars are nonzero constants. This is typical for a consistent system with a unique solution.

Question If we just wanted to know whether the linear system was consistent (i.e. whether a solution existed) how could we tell, without doing the whole computation? (Hint: look at the triangular form.)

Example 2 Determine whether the following system is consistent or inconsistent.

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & 6y & + & 3z & = & 0 \\ 4x & & & + & 2z & = & 24 \end{array}$$

Multiply the top equation by (-1), and add to the bottom equation. Then multiply the second equation by $(-1/3)$, and add to the bottom equation ...

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & 6y & + & 3z & = & 0 \\ & & & & 0 & = & 4 \end{array} \qquad \begin{pmatrix} 4 & -2 & 1 & 20 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

We can tell from the triangular form that this is inconsistent.

The operations on linear systems that we have been doing correspond to operations on the corresponding augmented matrix. They are called **elementary row operations**, and they are as follows:

1. Interchange two rows
2. Scale a row by a nonzero constant
3. Replacement: add to one row an multiple of another row.

Ask yourself: What is happening when we execute this procedure?