1 Overview

Main ideas:

1. definition of matrix-vector product as linear combination of columns
2. matrix-vector equation representation of a linear system
3. consistency of matrix-vector equation $Ax = b$ for all possible target vectors b and equivalent ways of thinking about this
4. row-vector rule for computing matrix-vector product
5. properties of the matrix-vector product (linearity)

Examples in text:

1. compute matrix-vector product using definition
2. write linear combination as matrix-vector product
3. compute matrix-vector product and notice pattern
4. compute matrix-vector product using row-vector rule

2 The Matrix-vector Equation Viewpoint

Recall from Section 1.3 that we can rewrite a linear system as a vector equation,

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & 6y & + & 3z & = & 0 \\ -12x & + & 6y & + & 2z & = & -40 \end{array} \quad \longleftrightarrow \quad x\begin{pmatrix} 4 \\ 0 \\ -12 \end{pmatrix} + y\begin{pmatrix} -2 \\ 6 \\ 6 \end{pmatrix} + z\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \\ -40 \end{pmatrix}$$

Now we *define* the product of an $m \times n$ matrix A with a vector v in $\mathbb{R}^n$ as the linear combination of the columns of A with weights given by the entries of v. This allows us to rewrite the left hand side of the vector equation as a matrix-vector product:

$$x\begin{pmatrix} 4 \\ 0 \\ -12 \end{pmatrix} + y\begin{pmatrix} -2 \\ 6 \\ 6 \end{pmatrix} + z\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 1 \\ 0 & 6 & 3 \\ -12 & 6 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Thus, we can rewrite a linear system as a matrix-vector equation,

$$\begin{array}{rcrcrcr} 4x & - & 2y & + & z & = & 20 \\ & & 6y & + & 3z & = & 0 \\ -12x & + & 6y & + & 2z & = & -40 \end{array} \quad \longleftrightarrow \quad \begin{pmatrix} 4 & -2 & 1 \\ 0 & 6 & 3 \\ -12 & 6 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \\ -40 \end{pmatrix}$$

Thus, finding a solution to a linear system is equivalent to expressing a given vector b (the last column in the augmented matrix) as a linear combination of a specified set of vectors (the columns of the coefficient matrix A), which is equivalent to finding a vector v such that the matrix-vector product Av equals the given vector b.

Further, asking whether a linear system is consistent is equivalent to asking whether a given vector b (the last column in the augmented matrix) lies in the span of the columns of the coefficient matrix A, which is equivalent to asking whether there is a solution vector v to the matrix equation $Av = b$.

3 Computing a Matrix-vector Product

Example 1 Compute the following matrix-vector product, using the definition above.

$$\begin{pmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

We compute the linear combination of the columns of the matrix using the weights in the vector:

$$\begin{pmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$$

Notice the sizes of the matrix and the vector. What would happen if you tried to compute

$$\begin{pmatrix} 1 & 3 \\ 3 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad ?$$

Example 2 Compute the following matrix-vector product, using the definition above.

$$\begin{pmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Again, we compute the linear combination of the columns of the matrix using the weights in the vector:

$$\begin{pmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} 3 \\ 2 \end{pmatrix} + z \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ 3x \end{pmatrix} + \begin{pmatrix} 3y \\ 2y \end{pmatrix} + \begin{pmatrix} -4z \\ z \end{pmatrix} = \begin{pmatrix} x + 3y - 4z \\ 3x + 2y + z \end{pmatrix}$$

What do you notice about the coefficients of x, y, and z in the final vector?

Question What is the geometric interpretation of the matrix-vector product? How is the vector Av related to the vector v geometrically? We will not answer this question right away, but it is worth keeping in the back of your mind.

4 Algebraic Properties of Vectors and Matrices

Worth mentioning, perhaps, are the following properties, which follow from the properties of the real numbers. See the shaded boxes on pages 27 (1.3) and 39 (1.4).

1. Vector addition: commutativity, associativity, zero element, additive inverses
2. Scalar multiplication: two "distributive" laws, associativity, identity
3. Matrix-vector product: linearity

These properties are exactly what we would expect/hope for; it is worth mentioning them, because later on we will adopt an abstract viewpoint, using these properties to define an abstract vector space and an abstract linear transformation.

5 Another existence question

In section 1.1 we discussed the question, Given a linear system, does a solution exist? Since then we have discussed this question from the vector-equation perspective and the matrix-vector equation perspective:

- Given a vector b and a set of vectors $\{a_1, \ldots, a_n\}$, is b in the span of $\{a_1, \ldots, a_n\}$?
- Given a matrix A and a vector b, is there a vector v such that $Av = b$?

Recall that in the 2-dimensional examples we discussed in class last time, one set of vectors spanned all of $\mathbb{R}^2$ and the other only spanned a line in $\mathbb{R}^2$. In general we might wonder, if we have two vectors in $\mathbb{R}^2$, do they span all of $\mathbb{R}^2$? Or, if we have three vectors in $\mathbb{R}^3$, do they span all of $\mathbb{R}^3$ (as opposed to just one point, one line, or one plane)?

This leads to the following question, rephrased from each perspective we have discussed:

- If we fix the coefficients of a linear system, is it true that there is a solution for *all possible* choices of $b_1, \ldots, b_n$, where $b_1, \ldots, b_n$ are the entries of the last column of the corresponding augmented matrix?
- Given a set of vectors $\{a_1, \ldots, a_n\}$ in $\mathbb{R}^m$, is it true that *all possible* vectors b in $\mathbb{R}^m$ lie in the span of $\{a_1, \ldots, a_n\}$?
- Given an m by n matrix A, is it true that for *all possible* vectors b in $\mathbb{R}^m$, there exists a vector v in $\mathbb{R}^n$ such that $Av = b$?

Example Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{pmatrix}$$

How many rows of A contain a pivot position? Does the equation $Ax = b$ have a solution for each b in $\mathbb{R}^4$?

Row reduction of A yields

$$\begin{pmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{pmatrix} \xrightarrow{\text{add } -2R_1 \text{ to } R_4} \cdots \longrightarrow \begin{pmatrix} 1 & 0 & -11 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, only three rows have pivot positions.

Suppose b is an arbitrary vector in $\mathbb{R}^4$. We want to determine whether there exists a vector v such that $Av = b$. We row reduce the augmented matrix:

$$\begin{pmatrix} 1 & 4 & 1 & 2 & b_1 \\ 0 & 1 & 3 & -4 & b_2 \\ 0 & 2 & 6 & 7 & b_3 \\ 2 & 9 & 5 & -7 & b_4 \end{pmatrix} \xrightarrow{\text{add } -2R_1 \text{ to } R_4} \cdots \longrightarrow \begin{pmatrix} 1 & 0 & -11 & 0 & b_1 - (2/15)(b_3 - 2b_2) - 4(b_2 + (4/15)(b_3 - 2b_2) \\ 0 & 1 & 3 & 0 & b_2 + (4/15)(b_3 - 2b_2) \\ 0 & 0 & 0 & 1 & (1/15)(b_3 - 2b_2) \\ 0 & 0 & 0 & 0 & -(1/7)(b_4 - 2b_1 - b_2) - (1/15)(b_3 - 2b_2) \end{pmatrix}$$

This is rather nasty, if I do say so myself. (And you have to wonder whether all the arithmetic that I didn't show you is actually correct ...) Are we any closer to answering the question? We want to see whether, for arbitrary $b_1, \ldots, b_4$, the corresponding linear system has a solution. It is apparent, by looking at the last row, that the system will only be consistent, if

$$-(1/5)(b_4 - 2b_1 - b_2) - (1/15)(b_3 - 2b_2) \quad = \quad 0 \qquad \text{i.e.} \qquad (1/105)(30b_1 + 29b_2 - 7b_3 - 15b_4) \quad = \quad 0$$

So, no, it will not be the case that the system will be consistent for arbitrary $b_1, \ldots, b_4$. Said another way, it is not true that the equation $Ax = b$ has a solution for each b in $\mathbb{R}^4$.

Notice that we could have answered the question right after we noticed that there are only three pivot rows.

Notice that we also have a description of the vectors b in $\mathbb{R}^4$ for which $Av = b$ has a solution. (Said another way, we have a description of the span of the columns of A.) This will become important later.